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An algebraic/numerical formalism for one-loop multi-leg amplitudes

T. Binoth[a], J.-Ph. Guillet[b], G. Heinrich[c], E. Pilon[b], C. Schubert[d]

[a] *Institut für Theoretische Physik, Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany*

[b] *Laboratoire d'Annecy-le-Vieux de Physique Théorique LAPTH, B.P. 110, F-74941 Annecy-le-Vieux Cedex, France*

[c] *Institut für Theoretische Physik, Universität Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland*

[d] *Department of Physics and Geology, University of Texas Pan American, 1201 West University Drive, Edinburg, Texas 78541, USA*

Abstract

We present a formalism for the calculation of multi-particle one-loop amplitudes, valid for an arbitrary number N of external legs, and for massive as well as massless particles. A new method for the tensor reduction is suggested which naturally isolates infrared divergences by construction. We prove that for $N \geq 5$, higher dimensional integrals can be avoided. We derive many useful relations which allow for algebraic simplifications of one-loop amplitudes. We introduce a form factor representation of tensor integrals which contains no inverse Gram determinants by choosing a convenient set of basis integrals. For the evaluation of these basis integrals we propose two methods: An evaluation based on the analytical representation, which is fast and accurate away from exceptional kinematical configurations, and a robust numerical one, based on multi-dimensional contour deformation. The formalism can be implemented straightforwardly into a computer program to calculate next-to-leading order corrections to multi-particle processes in a largely automated way.

Contents

1 Introduction

The quest for new physics at TeV colliders like the Fermilab Tevatron, the upcoming Large Hadron Collider at CERN, as well as the International Linear Collider (ILC) project, requires the quantitative calculation of many hard multi-particle processes. Direct searches rely on the proper identification of multi-particle signals and a precise understanding and determination of the corresponding multi-particle/multi-jet backgrounds. The latter, in particular those for multi-jet or vector boson(s) plus multi-jet production, are considerable, as can be estimated from leading order studies [1, 2]. Indirect searches at high luminosity machines will also involve high precision measurements of multi-particle observables of the Standard Model, comprising six-point processes like $e^+e^- \to 4$ fermions at the ILC, so that accurate predictions for these cross sections will be mandatory. High energy physics is thus entering a new era where the quantitative description of hard multi-particle production is becoming a topic of prominent importance, whereas the lowest order estimates for such processes are plagued by the well-known deficiencies of large renormalisation and factorisation scale dependences, poor multi-jet modelling and large sensibility to kinematic cuts. Therefore the calculation of next-to-leading-order (NLO) corrections to such cross sections is a necessary step forward. However, this task involves an enormous technical complexity.

To perform an NLO calculation with N hard external particles, the following points have to be addressed:

1. Generation of tree amplitudes with $N + 1$ external particles
2. Subtraction of soft and collinear singularities due to single unresolved real radiation
3. Generation of the one-loop amplitude with N external particles
4. Evaluation of the loop diagrams, UV renormalisation, extraction of soft and collinear singularities
5. Combination of the contributions above, cancellation of soft singularities, cancellation of collinear singularities or absorption into distribution functions
6. Numerical evaluation of the finite amplitude

For step 1, efficient, highly automated tools and algorithms are available. A similar statement can be made for step 2. Although automatisation is less trivial for this point, the available algorithms [3, 4, 5, 6] are well tested and in principle valid for an arbitrary number of legs. The same is not true for loop amplitudes. Although efficient programs like FeynArts [7], GRACE [8, 9] or QGRAF [10] exist which reliably deal with the combinatorial complexity of generating multi-leg one-loop Feynman diagrams, the evaluation of these diagrams, in particular in the presence of infrared divergences and for more than four external legs, is still far from being automated. So the bottleneck in constructing an automated program package for NLO cross sections is step 3, the evaluation of the loop graphs. Although a number of five-parton processes, see for example [11]–[31], have been calculated already, and very recently even physical $2 \to 4$ results have become available [32, 33], these calculations all required a tedious

individual treatment, and most of them do not allow to obtain fully differential results. Therefore it is desirable to have tools which allow the calculation of NLO cross sections in a largely automated way. These tools should be able to handle massless as well as massive particles, and should be numerically reliable and fast. Ideally, they should also allow to be interfaced with a parton shower in a universal way, using for example the formalism proposed in [34] or [35].

Several approaches to this aim have been suggested in the literature so far, from purely numerical ones to ones where the emphasis is on algebraic manipulations. A completely numerical approach has been worked out by D. Soper et al. [36, 37], where the sum over cuts for a given graph is performed before the numerical integration over the loop momenta. In this way unitarity is exploited to cancel soft and collinear divergences before they show up as explicit poles.

However, the conventional method of calculating the virtual (loop) and real (radiation) parts separately, thus generating infrared poles which cancel in the sum, is still the most widely and successfully used approach so far. Of course, within this approach, there are still many different ways to proceed, in particular in what concerns the evaluation of the one-loop amplitude. The most straightforward procedure – and historically the first one – relies on the use of recursion relations to reduce the tensor integrals occurring in the one-loop amplitude to a set of known basis integrals [38]–[57]. In the recent work on this subject, the emphasis is primarily on methods which are suitable for an efficient numerical evaluation of multi-leg amplitudes. For the massless case, a formalism has been proposed recently in [54], which produces spurious inverse Gram determinants, but in [55] a method is proposed how to deal with them. The formalism given in [51] avoids inverse Gram determinants in the reduction of pentagon integrals, but deals with massive particles only. In [56], another algorithm is presented, using spinor helicity methods. Based on the formalism of [56], an evaluation of one-loop integrals in massless gauge theories for up to 12 external legs has been given recently in [57].

A numerical approach to the one-loop integrals is the one of [58], where various concepts like the Bernstein-Tkachov theorem, Mellin-Barnes representation and sector decomposition are combined to get a stable numerical behaviour in all regions of configuration space. A fully numerical approach to the calculation of loop integrals by contour integration also has been elaborated in [59]. A semi-numerical approach, where a subtraction formalism for the UV and soft/collinear divergences of the one-loop graphs has been worked out, is presented in [60]. The idea is to integrate the remaining finite part in loop momentum space without performing any tensor reduction. Another semi-numerical approach is the one described in [61]. It relies on the fact that *every* one-loop amplitude can be represented in terms of building blocks which are one- and two-dimensional parameter integrals in a form which is suitable for numerical integration.

An alternative method is to obtain loop amplitudes by using unitarity to sew together tree amplitudes [62, 63, 64]. A difficulty of this approach has been to determine ambiguities of rational functions which are present when calculating QCD amplitudes. However, the application of twistor-space inspired methods to one-loop amplitudes [65] led to new insights in this context [66]–[70], and a rapid development in this direction may be expected in the future.

Nevertheless, to calculate one-loop amplitudes involving massive particles, as well as to

handle infrared divergences due to massless particles, we still have to rely on more conventional methods. As the size of the expressions for such amplitudes increases factorially with the number of legs, efficient methods of tensor reduction become more and more important. Although many in principle viable approaches exist, computations which rely on conventional reduction methods may get stuck due to the combinatorial growth of intermediate expressions for moderate values of $N\,(N \sim 6)$ already. It is the sheer size of the expressions, together with spurious denominators, which in the end hampers a successful, i.e. numerically stable, evaluation of the amplitude. Reduction algorithms in momentum space generically lead to so-called inverse Gram determinants which vanish if an exceptional kinematical configuration is approached. Reduction algorithms in Feynman parameter space in principle overcome this problem, but other kinematical determinants are still present in the denominator and one has to deal with scalar integrals in higher dimensions [42, 44, 45, 46, 54].

In this article, we propose an algorithm which is similar to the one given in [50], but improved in several respects. First, the new algorithm is designed to restrict and control the occurrence of inverse Gram determinants. Second, the formalism is valid for massive as well as massless particles, the soft and collinear divergences being regulated by dimensional regularisation. Our method is valid for arbitrary N, and we give a constructive recipe how to deal with the cases where kinematic matrices are not invertible. In addition, we prove explicitly in this formalism how N-point integrals with $N \geq 5$ in more than $n = 4-2\epsilon$ dimensions drop out of any physical one-loop amplitude. Moreover, we elaborate on the numerical evaluation of the basis integrals. Further, the new method is formulated in a manifestly shift invariant way and thus avoids a proliferation of terms due to shifts of the loop momentum when the tensor reduction is applied iteratively.

The outline of the paper is as follows. In section 2, we present a non-technical overview of our approach, which serves to point out its main features. The following sections contain a detailed description of the formalism. In section 3, we define our notation and the general setup. The method of tensor reduction by subtraction is described in section 4. In section 5 we elaborate the algebraic evaluation of the building blocks of our reduction. The case $N = 5$ is particularly interesting, and in section 6 we give form factors for $N = 5$ which do not contain higher dimensional 5-point functions *and* are free from inverse Gram determinants. The explicit proof that these integrals drop out and how the inverse Gram determinants cancel for $N = 5$ is rather technical and is provided in appendix C. In section 7 we deal with the numerical evaluation of the basis integrals, by means of multi-dimensional contour deformation, and we present explicit checks of the numerical stability near exceptional kinematical situations. Section 8 contains guidelines for the practitioner who is less interested in the mathematical details on how to implement the formalism directly into a computer code, before we conclude in section 9. In the appendices, we provide explicit formulae and useful relations for the direct application of our algorithm to multi-leg calculations.

2 A brief overview of the method

Before entering into the mathematical details of our formalism we would like to give a short overview of the method.

We consider one-loop N-point diagrams with external momenta $p_1, \ldots, p_N$. They are typically expressed in terms of integrals in momentum space, with and without loop momenta in the numerator. Any algebraic approach to evaluate these diagrams starts by reducing these tensor and scalar integrals to simpler objects, with the price to pay that the number of terms increases at each reduction step. We will distinguish reduction formulae for *tensor* and *scalar* integrals in the following.

Before applying tensor reduction formulae which typically increase the complexity of an expression, *reducible* terms might be cancelled. Numerators of Feynman integrals which contain scalar products between loop momenta and external momenta are called reducible if they can be expressed by differences of inverse propagators of the given Feynman diagram and by kinematical invariants. The momentum representations of the remaining irreducible tensor integrals are converted to linear combinations of form factors and Lorentz structures. In our approach this is done in a *non-standard* way, as we express the numerators of the tensor integrals in terms of propagator momenta instead of the loop momentum solely. Thus we define a generalised rank r tensor integral by

$$I_N^{n,\mu_1\ldots\mu_r}(a_1,\ldots,a_r) = \int \frac{d^nk}{i\,\pi^{n/2}}\,\frac{q_{a_1}^{\mu_1}\,\ldots q_{a_r}^{\mu_r}}{(q_1^2-m_1^2+i\delta)\ldots(q_N^2-m_N^2+i\delta)} \tag{1}$$

where $q_a = k + r_a$, and r_a is a combination of external momenta. The method is defined in $n = 4 - 2\epsilon$ dimensions and thus is applicable to general scattering processes with arbitrary propagator masses. Taking integrals of the form (1) as building blocks has two advantages: 1) combinations of loop and external momenta appear naturally in Feynman rules, 2) it allows for a formulation of the tensor reduction which manifestly maintains the invariance of the integral under a shift $k \to k + r_0$ in the loop momentum. Such a shift can be absorbed into a redefinition of the r_j, $r_j \to r_j - r_0$. The Lorentz structure of the integral (1) is carried by tensor products of the metric $g^{\mu\nu}$ and the difference vectors $\Delta_{ij}^\mu = r_i^\mu - r_j^\mu$, which are invariant under such a shift. The fact that the sums $q_a = k + r_a$, which generically appear in loop diagrams, are not split into loop and external momenta, as well as the explicit shift invariance, have the virtue of leading to a reduced number of terms in the expressions for the loop graphs.

A key point of our method is to reduce these tensor integrals by adding and subtracting terms such that a reduction into infrared (IR) *finite* and *reduced* (or pinched) integrals is achieved. By applying the reduction again to the potentially IR divergent reduced terms one generates iteratively a separation into IR finite and IR divergent terms. The latter are IR divergent 3-point functions which can be evaluated analytically in a closed form and separated from the finite part of the amplitude. This immediately provides a starting point for a numerical evaluation.

We further show that our reduction of N-point tensor integrals for $N \geq 6$ trivially maps to 5-point tensor integrals. The 5-point case needs special care, as was noted earlier [46]. We will show that all tensor 5-point functions can be reduced to some basis integrals without generating higher dimensional 5-point functions nor inverse Gram determinants. The form factors for all nontrivial tensor structures for up to rank five 5-point functions will be provided explicitly in terms of our basis integrals.

These basis integrals, i.e. the endpoints of our reduction, are 4-point functions in 6 dimensions

I_4^6, which are IR and UV finite, UV divergent 4-point functions in $n+4$ dimensions, and various 3-point functions, some of them with Feynman parameters in the numerator. This provides us with a very convenient separation of IR/UV divergences, as the IR poles are exclusively contained in the triangle functions. Explicitly, our reduction basis is given by integrals of the type

$$\begin{aligned}
I_3^n(j_1,\ldots,j_r) &= -\Gamma\left(3-\frac{n}{2}\right)\int_0^1 \prod_{i=1}^3 dz_i\,\delta(1-\sum_{l=1}^3 z_l)\,\frac{z_{j_1}\ldots z_{j_r}}{(-\frac{1}{2}\,z\cdot\mathcal{S}\cdot z-i\delta)^{3-n/2}}\,,\\
I_3^{n+2}(j_1) &= -\Gamma\left(2-\frac{n}{2}\right)\int_0^1 \prod_{i=1}^3 dz_i\,\delta(1-\sum_{l=1}^3 z_l)\,\frac{z_{j_1}}{(-\frac{1}{2}\,z\cdot\mathcal{S}\cdot z-i\delta)^{2-n/2}}\\
I_4^{n+2}(j_1,\ldots,j_r) &= \Gamma\left(3-\frac{n}{2}\right)\int_0^1 \prod_{i=1}^4 dz_i\,\delta(1-\sum_{l=1}^4 z_l)\,\frac{z_{j_1}\ldots z_{j_r}}{(-\frac{1}{2}\,z\cdot\mathcal{S}\cdot z-i\delta)^{3-n/2}}\,,\\
I_4^{n+4}(j_1) &= \Gamma\left(2-\frac{n}{2}\right)\int_0^1 \prod_{i=1}^4 dz_i\,\delta(1-\sum_{l=1}^4 z_l)\,\frac{z_{j_1}}{(-\frac{1}{2}\,z\cdot\mathcal{S}\cdot z-i\delta)^{2-n/2}}\,,
\end{aligned}\tag{2}$$

and $I_3^n, I_3^{n+2}, I_4^{n+2}, I_4^{n+4}$ with no Feynman parameters in the numerator. Of course, 2-point functions also have to be considered.

It turns out that for an arbitrary N-point amplitude, calculated in a gauge[1] where the rank can only be less or equal to the number of external legs, it is sufficient to consider $r \leq 3$ in the integrals above. Note that the $(n+2)$-dimensional box integrals, being neither IR nor UV divergent, can be evaluated using $n = 4$. The IR divergent three-point integrals are easy to handle, and we will give a complete list of all required 3-point integrals for the case of massless propagators in appendix B.

Our reduction formalism is designed such that any N-point amplitude can be written as a linear combination of the basis integrals without encountering inverse Gram determinants. We would like to emphasize that we do not only avoid Gram determinants of rank *four* matrices in the reduction of five-point functions, but obtain form factors which do not have inverse Gram determinants from *lower rank* matrices either. Our form factors are completely free from *any* inverse Gram determinants. In addition, we avoid the proliferation of higher dimensional integrals by choosing a convenient set of basic functions, while controlling at the same time the occurrence of arbitrary inverse Gram determinants. These two restrictions define our form factor representation and lead to the function basis we use. To our best knowledge no explicit form factor representations with these special properties have been derived in the literature before. To evaluate the basis functions, we propose here two complementary approaches: a purely numerical one and an algebraic one.

[1]Here we have in mind a gauge fixing for which gauge boson propagators are proportional to $g_{\mu\nu}$. In more general covariant gauges where the tensor structure of gauge boson propagators depends on the vector boson momentum, the extra gauge dependent terms might lead to higher dimensional integrals with non-trivial numerators. However, using the pinch technique [76], it can be shown that a reorganisation of the integrands allows to cancel the gauge dependent contributions at the level of the integrands before the loop momentum integration is performed, so that this complication can be avoided in these cases, too.

In the algebraic approach our basic buildings blocks, given in eq. (2), are further reduced to scalar integrals using recursion formulae. However, this introduces scalar integrals of dimensions higher than $n+4$ [42, 44, 54]. Applying scalar reduction formulae, the latter can be remapped to $(n+2)$-dimensional scalar box integrals and n-dimensional scalar two- and three-point functions. Doing so, the price to pay is the occurrence of inverse Gram determinants. These dangerous denominators are spurious: the expressions can be organised in such a way that inverse Gram determinants are multiplied by linear combinations of scalar integrals which also vanish in the case of exceptional kinematics, as has been done for example in [14]. Without such a grouping of terms, cancellations between singular pieces in general pose numerical problems, and even respecting such a grouping of terms does not guarantee numerical stability.

If one aims for compact algebraic expressions of loop amplitudes, it is a good strategy to express the amplitude in terms of simple scalar integrals and to enforce compensations of Gram determinants algebraically. Experience shows that compact expressions can be achieved in this way (for examples where algebraic reduction was successfully applied, see [31, 72, 73]).

In order to simplify the amplitude representation in terms of finite basis functions and to enforce explicit cancellations of Gram determinants, it can be useful to exploit relations between determinants and sub-determinants (minors) of the kinematic matrix $\mathcal{S}$ present in eq. (2). We provide a collection of relations useful for this purpose in appendix D. Many of these relations also served to obtain compact and convenient form factors for the tensor integrals.

If the amplitude is too complex, the purely algebraic treatment becomes intractable and it is advantageous to avoid the introduction of Gram determinants from the start. For this case we give a prescription how to evaluate the integrals in eq. (2) without further reduction. The IR divergent integrals, being only 3-point integrals in our approach, can easily be handled analytically, as they are simple enough to allow for explicit representations of all possible cases. For the finite 3-point and the 4-point functions we propose a new numerical method which allows for a direct numerical evaluation. By analytic continuation in Feynman parameter space and an adequate multidimensional contour deformation we find a numerically stable integral representation of the basis functions. The method is described in detail in section 7.1. In section 7.2, we compare the two approaches of either using eqs. (2) as endpoints of the reduction and then proceed numerically, or reducing algebraically until only scalar integrals are reached. The comparison shows that near exceptional momentum configurations the purely numerical evaluation is stable, whereas the analytic implementation of the basis functions, containing inverse Gram determinants, is not. This is true although the terms were grouped such that the coefficients of inverse Gram determinants also vanish in the limit of exceptional kinematics. We are lead to the conclusion that if the compensation of inverse Gram determinants is not possible algebraically, a tensor reduction scheme which avoids them from the start is preferable.

Thus we propose a method where the form factors are such that their numerical evaluation in the kinematically dangerous phase space regions poses no problem. Away from exceptional phase space regions, analytical representations can be used safely. These two complementary approaches should guarantee a successful evaluation of very complex multi-leg processes.

3 Definitions and notation

3.1 Feynman parameter representations

We consider general one-loop N-point graphs as the one shown in Fig. 1. All external momenta

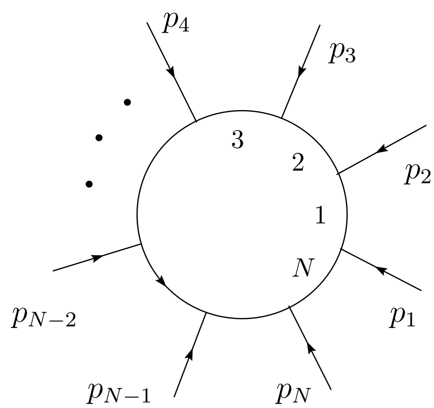

Figure 1: General N-point one-loop graph with momentum and propagator labelling.

p_i are defined as incoming. Momentum conservation implies

$$\sum_{i=1}^{N} p_i = 0 \; . \tag{3}$$

For future reference we label each propagator q_i by the number i as shown in Fig. 1. The *ordered* set containing the propagator labels is denoted by S. In Fig. 1 one has $S = \{1, 2, \ldots, N\}$.

The propagator or internal momenta are labelled accordingly by $q_i = k + r_i$, where k is the momentum running in the loop, and the momenta r_i are defined such that $p_i = r_i - r_{i-1}$, $(i = 1, \ldots, N)$, $, r_0 = r_N$. Thus one has $q_i = p_i + q_{i-1}$ $(q_0 = q_N)$. By momentum conservation, one can choose one of the vectors r_i to be zero. Most reduction algorithms specify either r_N or r_1 to be zero.

The momentum representation of the scalar N-point integral in n dimensions is denoted by

$$I_N^n(S) \;\; = \;\; \int \frac{d^n k}{i\pi^{n/2}} \, \frac{1}{\prod_{i=1}^{N}(q_i^2 - m_i^2 + i\delta)} \; . \tag{4}$$

In the following, we will use the shorthand notation $d\bar{k} = d^n k / i\pi^{n/2}$ for the integration measure. The ordered set S appearing here as an argument uniquely defines the one-loop integral. We will use the set S as a basic object throughout the paper.

After having introduced Feynman parameters and performed the momentum integration, $I_N^n(S)$ can be written as

$$I_N^n(S) \;\; = \;\; (-1)^N \Gamma(N - \frac{n}{2}) \int \prod_{i=1}^{N} dz_i \, \delta(1 - \sum_{l=1}^{N} z_l) \, \left(R^2\right)^{\frac{n}{2} - N}$$

$$R^2 = -\frac{1}{2}\, z \cdot \mathcal{S} \cdot z - i\delta = -\frac{1}{2} \sum_{i,j=1}^{N} z_i\, \mathcal{S}_{ij} z_j \ - i\delta \ . \tag{5}$$

The kinematic matrix $\mathcal{S}$ is defined by

$$\mathcal{S}_{ij} = (r_i - r_j)^2 - m_i^2 - m_j^2 \ . \tag{6}$$

In general, a one-loop N-point amplitude will contain N-point integrals as well as $(N-1), (N-2), \ldots, (N-M)$-point integrals with tree graphs attached to some of the external legs of the loop integral. The latter are characterised by the omission of some propagators (say $j_1, \ldots, j_m$) of the "maximal" one loop N-point graph. They consist of N external particles and $M < N$ internal lines, where M denotes the number of elements in the set $S \setminus \{j_1, \ldots, j_m\}$. We give some examples in Fig. 2. The corresponding kinematic matrix is denoted by

$$\mathcal{S}^{\{j_1 \cdots j_m\}} \equiv \mathcal{S}(S \setminus \{j_1, \ldots, j_m\}) \ . \tag{7}$$

It is obtained from $\mathcal{S}$ by replacing the entries of the rows and columns $j_1, \ldots, j_m$ by zero.

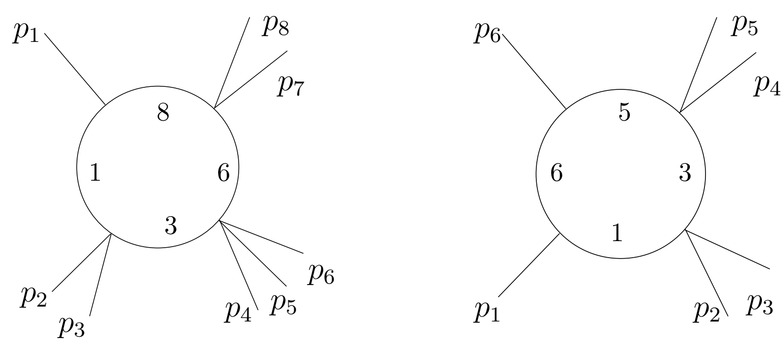

Figure 2: Graphical representation of pinch integrals. Each topology defines an ordered set S. The two diagrams correspond to $N = 8$, $M = 4$, $S = \{1, 3, 6, 8\}$ (left), and $N = 6$, $M = 4$, $S = \{1, 3, 5, 6\}$ (right).

In this way one can keep track of the pinching of propagators in the iterative application of reduction formulae without changing the labels of the rows and columns of reduced matrices $\mathcal{S}^{\{j_1 \cdots j_m\}}$ with respect to the maximal set S.

For example, for $S = \{1, 2, 3, 4\}$ one has, with $s_j = p_j^2$ and $s_{ij} = (p_i + p_j)^2$:

$$\mathcal{S} \ = \ \begin{pmatrix} -2\, m_1^2 & s_2 - m_1^2 - m_2^2 & s_{23} - m_1^2 - m_3^2 & s_1 - m_1^2 - m_4^2 \\ s_2 - m_1^2 - m_2^2 & -2\, m_2^2 & s_3 - m_2^2 - m_3^2 & s_{12} - m_2^2 - m_4^2 \\ s_{23} - m_1^2 - m_3^2 & s_3 - m_2^2 - m_3^2 & -2\, m_3^2 & s_4 - m_3^2 - m_4^2 \\ s_1 - m_1^2 - m_4^2 & s_{12} - m_2^2 - m_4^2 & s_4 - m_3^2 - m_4^2 & -2\, m_4^2 \end{pmatrix} \tag{8}$$

The symmetric (4×4) matrix $\mathcal{S}^{\{2,4\}}$, which corresponds to the pinching of propagators 2 and 4, is now defined by

$$\mathcal{S}^{\{2,4\}} \ = \ \begin{pmatrix} -2\, m_1^2 & 0 & s_{23} - m_1^2 - m_3^2 & 0 \\ 0 & 0 & 0 & 0 \\ s_{23} - m_1^2 - m_3^2 & 0 & -2\, m_3^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

Inverse matrices are labelled analogously. Although $\mathcal{S}^{\{j_1\cdots j_m\}}$ in our definition is not a regular matrix it has a uniquely defined pseudo-inverse. We recall that the so-called Moore-Penrose generalised inverse $\mathcal{P}$ to a symmetric matrix $\mathcal{S}$ is uniquely defined by the properties [77, 78]

$$\mathcal{PSP} = \mathcal{P}\,,\ \mathcal{SPS} = \mathcal{S}\,,\ \mathcal{PS} = \mathcal{SP}. \tag{10}$$

This concept will also be used below. To construct the pseudo inverse here one simply has to invert the sub-matrix of $\mathcal{S}^{\{j_1\cdots j_m\}}$ with the zero rows and columns omitted and promote the result back to an $N\times N$ matrix by inserting zeros for the rows and columns $\{j_1,\ldots,j_m\}$. In our example one finds, with $\lambda(x,y,z)=x^2+y^2+z^2-2xy-2yz-2xz$:

$$\mathcal{S}^{\{2,4\}-1} \;=\; \frac{1}{\lambda(s_{23},m_1^2,m_3^2)}\begin{pmatrix} 2\,m_3^2 & 0 & s_{23}-m_1^2-m_3^2 & 0\\ 0 & 0 & 0 & 0\\ s_{23}-m_1^2-m_3^2 & 0 & 2\,m_1^2 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{11}$$

In the following, $\mathcal{S}^{-1}(S\setminus\{j_1,\ldots,j_m\}) = \mathcal{S}^{\{j_1,\ldots,j_m\}-1}$ has to be understood in this sense.

Using these conventions, Feynman parameter integrals with propagator pinches and Feynman parameters $z_{l_1}\ldots z_{l_r}$ in the numerators can be defined as

$$\begin{aligned} &I_N^n(l_1,\ldots,l_r;S\setminus\{j_1,\ldots,j_m\}) = \\ &(-1)^N\Gamma(N-\frac{n}{2})\int\prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\,\delta(z_{j_1})\ldots\delta(z_{j_m})z_{l_1}\ldots z_{l_r}\left(R^2\right)^{n/2-N} \end{aligned} \tag{12}$$

Whenever one index of the pinch set $\mathcal{Q}=\{j_1,\ldots,j_m\}$ coincides with one numerator index in the set $\mathcal{N}=\{l_1,\ldots,l_r\}$ this integral is trivially zero:

$$\text{If }\ \mathcal{N}\cap\mathcal{Q}\neq\{\}\quad\Rightarrow\quad I_N^n(\mathcal{N};S\setminus\mathcal{Q})=0\;. \tag{13}$$

The above conventions lead to simple expressions in our formalism and are designed for iteration purposes, as will become clear below.

3.2 Definition of form factors

Algebraic expressions of an amplitude typically consist of spinors and Dirac chains of momenta and polarisation vectors, depending on the process under consideration. The loop momenta which appear in the numerator of a Feynman diagram often come in the combination $q_i=k+r_i$, like for instance in fermion propagators. We keep this natural combination in our tensor reduction formalism. Therefore we define tensor integrals as

$$I_N^{n,\mu_1\ldots\mu_r}(a_1,\ldots,a_r;S) = \int d\bar{k}\,\frac{q_{a_1}^{\mu_1}\ldots q_{a_r}^{\mu_r}}{\prod_{i\in S}(q_i^2-m_i^2+i\delta)}\;. \tag{14}$$

By setting $a_1,\ldots,a_r=N$, and using momentum conservation to set $r_N=0$, we can always retrieve the commonly used form

$$I_N^{n,\mu_1\ldots\mu_r}(N,\ldots,N,S) = \int d\bar{k}\,\frac{k^{\mu_1}\ldots k^{\mu_r}}{\prod_{i\in S}(q_i^2-m_i^2+i\delta)}\;. \tag{15}$$

In this more "conventional" approach, one of the q_i^μ is specified to be k^μ, which singles out one propagator and defines a standard form. After a reduction step one obtains integrals which are not of standard type, such that a shift operation $k \to k + r_j$ is necessary to remap to the standard form, giving rise to 2^r terms for a rank r tensor integral. Our formulation, being manifestly translation invariant and thus making such shifts obsolete, avoids here a proliferation of terms.

As pointed out above, shifts of the loop momentum can be absorbed into a redefinition of the r_j^μ vectors. To achieve a manifestly translation invariant formulation, we need vectors which are invariant under shifts $r_j^\mu \to r_j^\mu + r_a^\mu$. This motivates the definition of the shift-invariant vector Δ_{ij}^μ:

$$\Delta_{ij}^\mu = r_i^\mu - r_j^\mu = q_i^\mu - q_j^\mu \tag{16}$$

Apart from metric tensors $g^{\mu\nu}$, the Lorentz structure of the integrals will be carried by these vectors.

As will become clear below, we have to distinguish the cases $N \leq 5$ and $N \geq 6$. In the case $N \leq 5$ we will express the different tensor structures in terms of metric tensors and difference vectors, Δ_{ij}^μ. Tensor integrals are expressible by linear combinations of such Lorentz tensors and form factors denoted by $A_{l_1 \cdots l_r}^{N,r}(S)$, $B_{l_1 \cdots}^{N,r}(S)$, $C_{\cdots}^{N,r}(S)$. $A^{N,r}$ is the coefficient of the Lorentz structure containing only difference vectors. $B^{N,r}$ belongs to exactly one metric tensor and $(r-2)$ Δ_{ij}^μ vectors, and $C^{N,r}$ is the coefficient of the Lorentz structure containing products of two metric tensors. Thus our form factors for $N \leq 5$ are defined by the formula

$$\begin{aligned} I_N^{n,\mu_1 \cdots \mu_r}(a_1, \ldots, a_r; S) = & \\ & \sum_{j_1 \cdots j_r \in S} \left[\Delta_{j_1 \cdot}^{\cdot} \cdots \Delta_{j_r \cdot}^{\cdot}\right]_{\{a_1 \cdots a_r\}}^{\{\mu_1 \cdots \mu_r\}} A_{j_1 \ldots, j_r}^{N,r}(S) \\ + & \sum_{j_1 \cdots j_{r-2} \in S} \left[g^{\cdot\cdot} \Delta_{j_1 \cdot}^{\cdot} \cdots \Delta_{j_{r-2} \cdot}^{\cdot}\right]_{\{a_1 \cdots a_r\}}^{\{\mu_1 \cdots \mu_r\}} B_{j_1 \ldots, j_{r-2}}^{N,r}(S) \\ + & \sum_{j_1 \cdots j_{r-4} \in S} \left[g^{\cdot\cdot} g^{\cdot\cdot} \Delta_{j_1 \cdot}^{\cdot} \cdots \Delta_{j_{r-4} \cdot}^{\cdot}\right]_{\{a_1 \cdots a_r\}}^{\{\mu_1 \cdots \mu_r\}} C_{j_1 \ldots, j_{r-4}}^{N,r}(S) \end{aligned} \tag{17}$$

where the distribution of the r Lorentz indices μ_i, and momentum labels a_i to the vectors $\Delta_{j\,a_i}^{\mu_i}$, denoted by $[\cdots]_{\{a_1 \cdots a_r\}}^{\{\mu_1 \cdots \mu_r\}}$ in eq. (17), is illustrated in the following equations (19) to (22).

$$I_N^n(S) = A^{N,0}(S) \tag{18}$$

$$I_N^{n,\mu_1}(a_1; S) = \sum_{l \in S} \Delta_{l\,a_1}^{\mu_1} A_l^{N,1}(S) \tag{19}$$

$$I_N^{n,\mu_1\mu_2}(a_1, a_2; S) = \sum_{l_1, l_2 \in S} \Delta_{l_1\,a_1}^{\mu_1} \Delta_{l_2\,a_2}^{\mu_2} A_{l_1\,l_2}^{N,2}(S) + g^{\mu_1\,\mu_2} B^{N,2}(S) \tag{20}$$

$$\begin{aligned} I_N^{n,\mu_1\mu_2\mu_3}(a_1, a_2, a_3; S) = & \sum_{l_1, l_2, l_3 \in S} \Delta_{l_1\,a_1}^{\mu_1} \Delta_{l_2\,a_2}^{\mu_2} \Delta_{l_3\,a_3}^{\mu_3} A_{l_1 l_2 l_3}^{N,3}(S) \\ & + \sum_{l \in S} \left(g^{\mu_1\mu_2} \Delta_{l\,a_3}^{\mu_3} + g^{\mu_1\mu_3} \Delta_{l\,a_2}^{\mu_2} + g^{\mu_2\mu_3} \Delta_{l\,a_1}^{\mu_1}\right) B_l^{N,3}(S) \end{aligned} \tag{21}$$

$$I_N^{n,\mu_1\mu_2\mu_3\mu_4}(a_1,a_2,a_3,a_4;S) = \sum_{l_1\ldots l_4\in S} \Delta^{\mu_1}_{l_1\,a_1}\,\Delta^{\mu_2}_{l_2\,a_2}\,\Delta^{\mu_3}_{l_3\,a_3}\,\Delta^{\mu_4}_{l_4\,a_4}\,A^{N,4}_{l_1l_2l_3l_4}(S)$$
$$+\sum_{l_1,l_2\in S}\big(g^{\mu_1\mu_2}\,\Delta^{\mu_3}_{l_1\,a_3}\,\Delta^{\mu_4}_{l_2\,a_4} + g^{\mu_1\mu_3}\,\Delta^{\mu_2}_{l_1\,a_2}\,\Delta^{\mu_4}_{l_2\,a_4} + g^{\mu_1\mu_4}\,\Delta^{\mu_2}_{l_1\,a_2}\,\Delta^{\mu_3}_{l_2\,a_3} + g^{\mu_2\mu_3}\,\Delta^{\mu_1}_{l_1\,a_1}\,\Delta^{\mu_4}_{l_2\,a_4}$$
$$+g^{\mu_2\mu_4}\,\Delta^{\mu_1}_{l_1\,a_1}\,\Delta^{\mu_3}_{l_2\,a_3} + g^{\mu_3\mu_4}\,\Delta^{\mu_1}_{l_1\,a_1}\,\Delta^{\mu_2}_{l_2\,a_2}\big)\;B^{N,4}_{l_1l_2}(S)$$
$$+\left(g^{\mu_1\mu_2}\,g^{\mu_3\mu_4} + g^{\mu_1\mu_3}\,g^{\mu_2\mu_4} + g^{\mu_2\mu_3}\,g^{\mu_1\mu_4}\right)\,C^{N,4}(S) \tag{22}$$

We recall that standard form factor representations can be simply obtained by replacing $a_j = N$ for all j, together with $r_N = 0$. This also shows that the form factors do not depend on the introduction of the difference vector. The form factors are shift invariant by themselves. One main result of the paper will be the explicit representation of all these form factors in terms of higher dimensional 4-point parameter integrals and $n = 4 - 2\epsilon$ dimensional 3-point parameter integrals with nontrivial numerators, see eq. (2).

For $N \geq 6$, the tensor reduction will be done in such a way that only the form factors for $N = 5$ appear. The Lorentz structure of N-point rank r tensor integrals does not require the introduction of additional factors of $g^{\mu\nu}$ as compared to the $N = 5$ case, only additional difference vectors Δ^{μ}_{ij} appear. This is due to the fact that for $N \geq 5$, four linearly independent external vectors form a basis of Minkowski space. We note that for $N = 5$, one could already express the metric by external momenta, but this would introduce inverse Gram determinants.

Before closing this section we would like to note that momentum integrals, Feynman parameter integrals and form factors are naturally related by [42, 44, 50]:

$$\begin{aligned} I_N^{n,\mu_1\cdots\mu_r}(a_1,\ldots,a_r\,;S) &= (-1)^r \sum_{m=0}^{[r/2]}\left(-\frac{1}{2}\right)^m \sum_{j_1\cdots j_{r-2m}=1}^{N} \left[(g^{\cdot\cdot})^{\otimes m}\,\Delta^{\cdot}_{j_1\cdot}\cdots\Delta^{\cdot}_{j_r\cdot}\right]^{\{\mu_1\cdots\mu_r\}}_{\{a_1\cdots a_r\}} \\ &\quad \times I_N^{n+2m}(j_1\ldots,j_{r-2m}\,;S) \end{aligned} \tag{23}$$

where $I_N^{n+2m}(j_1\ldots,j_{r-2m}\,;S)$ is defined in eq. (12). In eq. (23), $[r/2]$ stands for the nearest integer less or equal to $r/2$ and the symbol $\otimes m$ indicates that m powers of the metric tensor are present. It is obvious from this formula that the $g^{\mu\nu}$-terms are always associated to integrals in more than n dimensions.

In the following sections 4 and 5, we will formulate a reduction formalism for tensor integrals of the type (14) and give convenient representations for the form factors defined in eq. (17). In section C we will show that higher than $n = 4 - 2\epsilon$ dimensional integrals for $N \geq 5$ can be avoided completely if the external kinematics is defined in 4-dimensions, a fact which was already carefully investigated elsewhere [42, 46, 50].

4 Tensor reduction by subtraction

The formalism described in this section naturally leads to a separation of IR divergent and finite expressions and does not produce spurious Gram determinants. The reduction is based

on a subtraction technique which is analogous to the one used in [50] for the scalar case. Before we come to the tensorial case, let us recall the procedure of [50] for the scalar case, recast into the notation of this article.

4.1 Subtraction for scalar integrals

Our aim is to split a scalar N-point integral as defined in eq. (4) into an IR finite part and a possibly IR divergent, but simpler part. Therefore we make the ansatz

$$\begin{aligned} I_N^n(S) &= I_{div}(S) + I_{fin}(S) \\ &= \sum_{i\in S} b_i(S) \int d\bar{k}\, \frac{(q_i^2 - m_i^2)}{\prod_{j\in S}(q_j^2 - m_j^2 + i\,\delta)} + \int d\bar{k}\, \frac{1 - \sum_{i\in S} b_i(S)\,(q_i^2 - m_i^2)}{\prod_{j\in S}(q_j^2 - m_j^2 + i\,\delta)} \end{aligned} \tag{24}$$

We can see that I_{div} is a sum of reduced integrals where one propagator has been pinched. Now let us consider I_{fin} after having introduced Feynman parameters. In order to arrive at a quadratic form in the loop momentum, we perform the shift

$$k = l - \sum_{i\in S} z_i\, r_i \, . \tag{25}$$

This shift transforms the denominator to the form $l^2 - R^2$, where R^2 is defined in eq. (5). For the numerator, we use

$$\begin{aligned} \Delta_{ij}\cdot\Delta_{kl} &= \frac{1}{2}(\mathcal{S}_{il} + \mathcal{S}_{jk} - \mathcal{S}_{ik} - \mathcal{S}_{jl}) \, , \\ \left(\sum_{i\in S} z_i\,\Delta_{ji}\right)^2 &= \sum_{i\in S} z_i\,\mathcal{S}_{ij} + m_j^2 + R^2 \end{aligned} \tag{26}$$

to obtain

$$1 - \sum_{i\in S} b_i(S)\,(q_i^2 - m_i^2) = -(l^2 + R^2)\sum_{i\in S} b_i(S) + \sum_{j\in S} z_j \left[1 - \sum_{i\in S} b_i(S)\, \{\mathcal{S}_{ij} + 2\,l\cdot\Delta_{ij}\}\right] . \tag{27}$$

The term linear in the loop momentum l vanishes due to symmetric integration and we conclude that if the equation

$$\sum_{i\in S} b_i(S)\,\mathcal{S}_{ij} = 1 \quad , \quad j = 1,\ldots,N \tag{28}$$

is fulfilled, the term in square brackets in eq. (27) vanishes. Then I_{fin} is given by

$$\begin{aligned} I_{fin}(S) &= -B(S)\,\Gamma(N) \int_0^1 \prod_{i\in S} dz_i\,\delta(1 - \sum_{l\in S} z_l) \int \frac{d^n l}{i\pi^{n/2}}\, \frac{l^2 + R^2}{(l^2 - R^2)^N} \\ &= -B(S)\,(N - n - 1)\, I_N^{n+2}(S) \end{aligned} \tag{29}$$

$$B(S) = \sum_{i\in S} b_i(S) \, . \tag{30}$$

If it is clear from the context which set S we refer to, the argument (S) is omitted in $B(S)$ and $b_i(S)$. If the b_i belong to a reduced kinematic matrix $\mathcal{S}^{\{j\}}$ where the j^{th} row and column is zero, associated with the set $S \setminus \{j\}$, one has $b_i(S \setminus \{j\}) = \sum_{k\in S\setminus\{j\}} (\mathcal{S}^{\{j\}})^{-1}_{ki}$. For simplicity of notation, we introduce the shorthand $b_i(S \setminus \{j\}) = b_i^{\{j\}}$, and correspondingly $B^{\{j\}}$ is defined as

$$B^{\{j\}} = \sum_{i\in S\setminus\{j\}} b_i^{\{j\}} \; . \tag{31}$$

All that remains to be shown now is that eq. (28) indeed has a solution for the reduction coefficients b_i for arbitrary N.

In the case of non-exceptional 4-dimensional kinematics[2], $rank(\mathcal{S}) = min(N, 6)$ holds. Thus $\mathcal{S}$ is invertible for $N \leq 6$, and eq. (28) has the unique solution

$$b_i = \sum_{k\in S} \mathcal{S}^{-1}_{ki} \; . \tag{32}$$

If $\mathcal{S}$ is not invertible, we proceed as follows. First we single out the a^{th} row and column in $\mathcal{S}$, where a is an arbitrary element of the set S, to write $\mathcal{S}_{ij}$ as:

$$\mathcal{S}_{ij} = -G^{(a)}_{ij} + V^{(a)}_i + V^{(a)}_j \tag{33}$$

with

$$G^{(a)}_{ij} = 2\Delta_{ia}\cdot\Delta_{ja}, \quad V^{(a)}_i = \Delta^2_{ia} - m_i^2 \; . \tag{34}$$

The Gram matrix $G^{(a)}_{ij}$ is understood as an $N \times N$ matrix. By definition its entries are zero whenever $i = a$ or $j = a$. For ease of notation, we will omit the superscript (a) in the following if it is clear from the context. Using the above relations and distinguishing the cases $j = a$ and $j \neq a$, eq. (28) is equivalent to the two equations

$$\sum_{i\in S\setminus\{a\}} b_i\, G_{ij} \;=\; B\,(V_j - V_a) \tag{35}$$

$$\sum_{i\in S\setminus\{a\}} b_i\,(V_i - V_a) \;=\; 1 - 2\,B\,V_a \; . \tag{36}$$

Eqs. (35),(36) may be solved in the following way. As G is not a regular matrix we first construct the generalised inverse H of G, defined as in eq. (10). To this end we introduce four linearly independent 4-vectors $E^\mu_{l=1,\cdots,4}$ forming a basis of the physical Minkowski space, and a $4 \times N$ coefficient matrix $\mathcal{R}$, such that

$$\Delta^\mu_{i\,a} = \sum_{m=1}^{4} \mathcal{R}^{(a)}_{mi}\, E^\mu_m, \quad \tilde{G}_{mn} = 2E_m\cdot E_n \; . \tag{37}$$

[2] We call a kinematic configuration defined by N external momentum vectors *exceptional* if the Gram determinant built from a subset of four of these vectors vanishes.

Here $\mathcal{R}^{(a)}_{ma} = 0$ for all m. The Gram matrix G is expressible as[3] $G = \mathcal{R}^T \tilde{G} \mathcal{R}$. The matrix $\mathcal{R}$ is of rank 4, and thus the 4×4 matrix $\mathcal{R}\mathcal{R}^T$ is invertible. The sought matrix H is then uniquely defined by

$$H = \mathcal{R}^T (\mathcal{R}\mathcal{R}^T)^{-1} \tilde{G}^{-1} (\mathcal{R}\mathcal{R}^T)^{-1} \mathcal{R} \,. \tag{38}$$

Before going on we note that the case of exceptional kinematics is now easily dealt with. If the external vectors only form a space of dimension k less then four, one may choose k basis vectors $E^\mu_{l=1,\cdots,k}$ defining $\mathcal{R}$ to be a $k \times N$ matrix. This is the only change to be made.

The system of equations (35),(36) admits solutions if and only if the consistency condition

$$B \sum_{j\in S\setminus\{a\}} (\mathbb{1}_{N-1} - GH)_{ij} \, \delta V^{(a)}_j \;\; = \;\; 0 \tag{39}$$

is fulfilled, where we have defined

$$\delta V^{(a)}_j = V^{(a)}_j - V^{(a)}_a \quad , \quad j \in S \setminus \{a\} \,. \tag{40}$$

We will not always write the sum $\sum_{j\in S\setminus\{a\}}$ explicitly in the following, but denote products like $\sum_{j\in S\setminus\{a\}} K_{ij} \, V_j$ by $(K \cdot V)_i$.

Since, for $N > 5$, in general $GH \cdot \delta V \neq \delta V$, a solution of (39) for $N > 5$ exists if and only if $B = 0$. The solution spans an $(N-5)$-dimensional space which is just the kernel of the Gram matrix. It can be parametrized by $(N-5)$ vectors $U^{(1,\ldots,N-5)}$. Let us note that

$$GH = HG = \mathcal{R}^T (\mathcal{R}\mathcal{R}^T)^{-1} \mathcal{R} \tag{41}$$

and that the projector onto $\mathrm{Ker}(G)$ is given by

$$K \;\; = \;\; \mathbb{1}_{N-1} - \mathcal{R}^T (\mathcal{R}\mathcal{R}^T)^{-1} \mathcal{R} \,. \tag{42}$$

It follows from the definition of H that $K \cdot \delta V \in \mathrm{Ker}(G)$. Now one can choose $U^{(N-5)} = K \cdot \delta V / (\delta V \cdot K \cdot \delta V)$ parallel to δV and the other U-vectors orthogonal, $\delta V \cdot U^{(k=1,\ldots,N-6)} = 0$. A general vector in $\mathrm{Ker}(G)$ is then parametrised by $U = \sum_{k=1}^{N-6} \beta_k U^{(k)} + \alpha U^{(N-5)}$ and a solution to eqs. (35),(36) is given by

$$b_i \;\; = \;\; \frac{(K \cdot \delta V)_i + \sum_{k=1}^{N-6} \beta_k U^{(k)}_i}{\delta V \cdot K \cdot \delta V} \,, \quad i \in S \setminus \{a\} \tag{43}$$

$$b_a \;\; = \;\; - \sum_{k\in S\setminus\{a\}} b_k \,. \tag{44}$$

We recall that this solution is valid for all $N \geq 6$. For exceptional kinematics and $N < 6$ an analogous solution can be derived. In the other cases one can simply use eq. (32).

[3]Note that the pseudo-inverse of G and the singular value decomposition of G given in ref. [54] are related in a simple way [78]: Defining an orthogonal 4×4 matrix O which diagonalises $\tilde{G}_{mn}$ (such a matrix O always exists for a symmetric matrix $\tilde{G}$), i.e. $(O^T \tilde{G} O)_{mn} = \omega_m \delta_{mn}$, one has $G_{ij} = (\mathcal{R}^T \, O \, O^T \, \tilde{G} \, O \, O^T \, \mathcal{R})_{ij} = \sum_{k=1}^4 (\mathcal{R}^T \, O)_{ik} \, \omega_k \, (O^T \, \mathcal{R})_{kj} = \sum_{k=1}^4 u_{ik} \, \omega_k \, v^T_{kj}$, which means that the u_{ik} and v^T_{kj} used in ref. [54] are related to the matrix $\mathcal{R}$ given here by $u_{ik} = (\mathcal{R}^T O)_{ik}$ and $v^T_{kj} = (O^T \mathcal{R})_{kj}$.

Near some special momentum configurations the denominators of the reduction coefficients, $\delta V \cdot K \cdot \delta V$ or $\det(\mathcal{S})$, may cause numerical instabilities in realistic applications. In contrast to inverse Gram determinants which can be viewed as technical artefacts of the tensor reduction down to scalar integrals in momentum space, the singular behaviour of the latter is due to physical singularities like soft/collinear configurations or thresholds and therefore they are unavoidable. We would like to point out that this is not a problem related to the use of the pseudo-inverse. If one would use the singular value decomposition, this problem would occur as well. Experimental cuts very often exclude these numerically dangerous phase space regions.

Finally, we quote the interesting relation between the kinematic determinants

$$B \ \det \mathcal{S} = (-1)^{N+1} \det G^{(a)} \ , \tag{45}$$

where $G_{ij}^{(a)}$ is the Gram matrix defined above. The relation is valid for all $a \in S$ and arbitrary N.

We conclude that in the case $N \geq 6$ one always has $B = 0$, if the external kinematics is defined in four dimensions. This means that $I_{fin}(S) = 0$ in eq. (29). In the case $N = 5$ one finds $I_{fin}(S) = B\,(n-4)\, I_5^{n+2}(S)$. As $I_5^{n+2}(S)$ is IR and UV finite, the whole term is of $\mathcal{O}(\epsilon)$ and can be dropped in phenomenological applications. For $N = 4$ we see that 4-point functions can be represented in terms of 3-point functions and $(n+2)$-dimensional 4-point functions. As the latter are IR finite, a splitting into IR divergent and IR finite integrals is achieved. We would like to emphasise that by iteration of eqs. (24) and (29) an *arbitrary* scalar N-point function can be algebraically reduced to 3-point functions and $(n+2)$-dimensional 4-point functions. In this representation IR divergent integrals are naturally separated from finite contributions.

As an example we give here the 6-point function in terms of $(n+2)$-dimensional 4-point functions and 3-point functions. In our notation, up to $\mathcal{O}(\epsilon)$ terms:

$$I_6^n(S) \;=\; \sum_{j\in S} b_j \sum_{k\in S\setminus\{j\}} b_k^{\{j\}} \left[B^{\{j,k\}}\, I_4^{n+2}(S\setminus\{j,k\}) + \sum_{l\in S\setminus\{j,k\}} b_l^{\{j,k\}}\, I_3^n(S\setminus\{j,k,l\}) \right]$$

From this representation, IR divergences can be trivially isolated.

4.2 Subtraction for tensor integrals

We now extend the above reasoning to the tensorial case, i.e. we will split a general tensor integral into an infrared finite part and a part which contains possible IR divergences, the latter having one rank less and one propagator less. We will give a constructive algorithm how to proceed for any number N of external legs.

Let us write eq. (14) as follows:

$$\begin{aligned} I_N^{n,\mu_1\ldots\mu_r}(a_1,\ldots,a_r;S) \;=\; & \int d\bar{k}\, \frac{\left[q_{a_1}^{\mu_1} + \sum_{j\in S} \mathcal{C}_{ja_1}^{\mu_1}\,(q_j^2 - m_j^2)\right]\; q_{a_2}^{\mu_2}\ldots q_{a_r}^{\mu_r}}{\prod_{i\in S}(q_i^2 - m_i^2 + i\delta)} \\ & - \sum_{j\in S} \mathcal{C}_{ja_1}^{\mu_1} \int d\bar{k}\, \frac{(q_j^2 - m_j^2)\; q_{a_2}^{\mu_2}\ldots q_{a_r}^{\mu_{a_r}}}{\prod_{i\in S}(q_i^2 - m_i^2 + i\delta)} \ . \end{aligned} \tag{46}$$

The last line corresponds to $(N-1)$-point tensor integrals of rank $(r-1)$. The coefficients $\mathcal{C}^{\mu_1}_{ja_1}$ will be determined such that the first term in eq. (46) is an IR finite expression.

As in the scalar case, one introduces N Feynman parameters z_i and makes the substitution (25) to obtain the form $l^2 - R^2$ for the denominator. Under the shift (25), the momenta q_a become

$$q_a = l + \sum_{i\in S} z_i\, \Delta_{a\,i}\ . \tag{47}$$

Now let us consider the vector $A^{\mu}_{a_1}$, given by the square bracket in the first line of eq. (46)

$$A^{\mu_1}_{a_1} = q^{\mu_1}_{a_1} + \sum_{j\in S} \mathcal{C}^{\mu_1}_{ja_1}\,(q_j^2 - m_j^2) \tag{48}$$

where $\mathcal{C}^{\mu}_{ja}$ is defined by the following equation, which will be the cornerstone of our derivation for general N:

$$\sum_{j\in S} \mathcal{S}_{ij}\, \mathcal{C}^{\mu}_{j\,a} = \Delta^{\mu}_{i\,a}\ , \qquad a\in S\ . \tag{49}$$

Note that this equation is analogous to eq. (28) for the scalar case. We will solve equation (49) in the next subsection for arbitrary kinematics and arbitrary N.

The vector A^{μ}_{a} transforms under the shift (25) as:

$$\begin{aligned}
A^{\mu}_{a} &= l^{\mu} + \sum_{i\in S} z_i\, \Delta^{\mu}_{ai} + \sum_{j\in S} \mathcal{C}^{\mu}_{ja} \left[l^2 - m_j^2 + \left(\sum_{i\in S} z_i\, \Delta_{ji} \right)^2 - 2\, l\cdot \sum_{i\in S} z_i\, \Delta_{ij} \right] \\
&= l^{\mu} + \left(l^2 + R^2\right)\, \mathcal{V}^{\mu}_{a_1} + \sum_{i\in S} z_i \left[\sum_{j\in S} \mathcal{C}^{\mu}_{ja}\, \left(\mathcal{S}_{ij} - 2\, l\cdot\Delta_{ij}\right) - \Delta^{\mu}_{ia} \right]\ ,
\end{aligned} \tag{50}$$

where eq. (26) and the definition

$$\mathcal{V}^{\mu}_{a} = \sum_{j\in S} \mathcal{C}^{\mu}_{ja} = \sum_{k\in S} b_k\, \Delta^{\mu}_{k\,a} \tag{51}$$

have been used. Thus we see that if eq. (49) is fulfilled, all the terms contained in A^{μ}_{a} are either proportional to the loop momentum l or to R^2.

Provided that eq. (49) is fulfilled, and using $\Delta_{ij} = \Delta_{ia_2} + \Delta_{a_2 j}$, where $a_2 \in S$ is arbitrary, eq. (50) can also be written in the following form:

$$A^{\mu}_{a_1} = l_{\nu} \left(\mathcal{T}^{\mu\nu}_{a_1 a_2} + 2\, \mathcal{V}^{\mu}_{a_1} \sum_{i\in S} z_i \Delta^{\nu}_{a_2 i} \right) + \left(l^2 + R^2\right)\, \mathcal{V}^{\mu}_{a_1}\ , \tag{52}$$

where

$$\mathcal{T}^{\mu\nu}_{a_1\,a_2} = g^{\mu\nu} + 2 \sum_{j\in S} \mathcal{C}^{\mu}_{j\,a_1}\, \Delta^{\nu}_{j\,a_2}\ . \tag{53}$$

From the expression (52) one can see immediately that the first term on the right-hand side in eq. (46) has no infrared divergences: After the shift (25), the integral in the first line is proportional to l or $l^2 + R^2$ and, after integration over the loop momentum, will give some higher-dimensional integrals (with Feynman parameters in the numerator for $r > 1$). The term in the second line of eq. (46) is divergent but has one tensor rank less and one propagator less. Thus, if eq. (49) is fulfilled, the ansatz (46) leads to the desired splitting of a rank r N-point integral into IR finite terms and $(N-1)$-point integrals of rank $r-1$.

4.3 Solving the defining equation for arbitrary N

It remains to be shown that eq. (49) has a solution for general N, and to construct such a solution explicitly.

As in the scalar case, the solution is simple if $\mathcal{S}$ is invertible, i.e. in the case of non-exceptional kinematics for $N \leq 6$. In this case, eq. (49) has the unique solution

$$\mathcal{C}^{\mu}_{ja} = \sum_{k \in S} \left(\mathcal{S}^{-1}\right)_{jk} \Delta^{\mu}_{ka} \, . \tag{54}$$

On the other hand, if $N \geq 7$ or in the case of exceptional kinematics, $\mathcal{S}$ is not invertible, so eq. (49) does not have a unique solution. However, an explicit solution can be constructed in the same way as has been done in section 4.1 for the scalar case. To this end, we again write the matrix $\mathcal{S}$ as in eqs. (33),(34). Using the definition (40) for $\delta V_j^{(a)}$, eq. (49) may be rewritten as

$$\begin{aligned} \sum_{i \in S} \mathcal{S}_{ij} \, \mathcal{C}^{\mu}_{i\,b} &= \Delta^{\mu}_{j\,b} \, , \quad b \in S \qquad \Leftrightarrow \\ \sum_{i \in S \setminus \{a\}} G^{(a)}_{ij} \, \mathcal{C}^{\mu}_{i\,b} &= -\Delta^{\mu}_{j\,a} + \delta V_j^{(a)} \, \mathcal{V}^{\mu}_b , \qquad (55) \\ \sum_{i \in S \setminus \{a\}} \delta V_i^{(a)} \, \mathcal{C}^{\mu}_{i\,b} &= \Delta^{\mu}_{a\,b} - 2 V_a^{(a)} \, \mathcal{V}^{\mu}_b \qquad (56) \end{aligned}$$

where also $\sum_{i \in S} \mathcal{C}^{\mu}_{i\,b} = \mathcal{V}^{\mu}_b$ has been used. Eqs. (55),(56) can be solved using the same pseudo-inverse as already constructed in section 4.1. The system of equations (55),(56) admits solutions if and only if the consistency condition

$$\sum_{j \in S \setminus \{a\}} (\mathbb{1}_{N-1} - GH)_{ij} \left(\Delta^{\mu}_{j\,a} - \delta V_j^{(a)} \, \mathcal{V}^{\mu}_b \right) = 0 \tag{57}$$

is fulfilled. Since $\sum_{j \in S \setminus \{a\}} (\mathbb{1}_{N-1} - GH)_{ij} \Delta^{\mu}_{ja} = 0$ whereas, for $N > 5$, in general $GH \cdot \delta V \neq \delta V$, a solution of (55) for $N > 5$ exists if and only if $\mathcal{V}^{\mu}_b = 0$. The general solution of (55) for $N > 5$ is thus given by

$$\begin{aligned} \mathcal{C}^{\mu}_{i\,b} &= - \sum_{j \in S \setminus \{a\}} H_{ij} \, \Delta^{\mu}_{j\,a} + W^{\mu}_i \quad , \quad i \in S \setminus \{a\} \\ \mathcal{C}^{\mu}_{a\,b} &= - \sum_{j \in S \setminus \{a\}} \mathcal{C}^{\mu}_{j\,b} \quad , \quad \mathcal{V}^{\mu}_b = 0 \, . \end{aligned} \tag{58}$$

The vectors W_i^μ span the $(N-5)$-dimensional vector space Ker(G), the kernel of G. To construct a basis of Ker(G) we again use the vectors $U_i^{(l=1,\cdots N-5)}$ introduced in section 4.1. The vectors W_i^μ are then parametrised as

$$W_i^\mu = \beta^\mu_{N-5}(K\cdot\delta V)_i + \sum_{l=1}^{N-6}\beta_l^\mu\, U_i^{(l)} \;. \tag{59}$$

Substituting the parametrisation (59) into eq. (56) yields:

$$\begin{aligned}\beta^\mu_{N-5} &= \frac{1}{\delta V\cdot K\cdot\delta V}\left(\Delta^\mu_{ab} + \sum_{i,j\in S\setminus\{a\}}\delta V_i\, H_{ij}\,\Delta^\mu_{j\,a}\right) \\ \beta^\mu_{l=1,\cdots,N-6} &= \text{arbitrary 4-vectors}\;.\end{aligned} \tag{60}$$

If $N=6$, G is not invertible, yet the solution (58)-(60) is still unique, as it has to be since $\mathcal{S}$ is invertible. If $N\geq 7$, neither G nor $\mathcal{S}$ are invertible. In this case (49) still admits solutions, which are however no more unique, but span the $(N-6)$-dimensional affine space defined by (58)-(60). It also has to be emphasised that the above construction is equally valid if the external momenta become linearly dependent not due to the fact that $N\geq 6$, but due to an exceptional kinematic configuration.

With relations (37) and (58) at hand, it is now easy to see that for $N\geq 6$ one has

$$\sum_{j\in S}\mathcal{C}^\mu_{j\,b}\,\Delta^\nu_{j\,a} \;=\; -\frac{1}{2}\,g^{\mu\nu}_{[4]} \;\text{ and thus }\; \mathcal{T}_{[4]}{}^{\mu\,\nu}_{a\,b} = 0\;. \tag{61}$$

The subscript "[4]" in $\mathcal{T}_{[4]}{}^{\mu\,\nu}_{a\,b}$ indicates that $\mathcal{T}^{\mu\,\nu}_{a\,b}$ is 4-dimensional and not n-dimensional here. Further, we already saw that

$$\mathcal{V}^\mu_b = 0 \;\text{ for }\; N\geq 6\;. \tag{62}$$

Relations (61) and (62) have an important consequence. They mean that A^μ_a in eq. (52) is zero and thus they imply that no higher than n-dimensional N-point functions with $N\geq 6$ can be generated by the reduction: In eq. (46), only the pinched terms survive, leading to

$$I_N^{n,\mu_1\ldots\mu_r}(a_1,\ldots,a_r;S) \;=\; -\sum_{j\in S}\mathcal{C}^{\mu_1}_{ja_1}\, I_{N-1}^{n,\mu_2\ldots\mu_r}(a_2,\ldots,a_r;S\setminus\{j\}) \quad (N\geq 6)\;. \tag{63}$$

In this sense the tensor reduction of N-point integrals with $N\geq 6$ is trivial: Integrals with $N\geq 6$ can be reduced iteratively to 5-point integrals, without generating higher dimensional remainders. Therefore form factors for $N>5$ are not needed.

The case $N=5$ needs special attention and was already carefully analysed in the Feynman parameter approach [46]. We will give a full analysis of the 5-point case within our formalism in section C.

In order to make contact to ref. [50], we note that the reduction is equivalent to applying first eq. (32) and then eq. (B.7) of [50], but the method advocated here gives very naturally certain algebraic relations between reduction coefficients, which have been exploited in order to avoid Gram determinants (i.e. factors of $1/B$) as far as possible.

5 Form factors for $N = 3, 4$ and algebraic representation of basis integrals

In the following we will apply the formalism explained above to provide explicit formulae for all form factors present in the reduction of 3- and 4-point integrals. The form factors for $N = 5$ will be given in section 6 and the 2-point functions will be listed in appendix A. The form factors will contain scalar integrals with non-trivial numerators. In this representation no inverse Gram determinants are present, which serves as an ideal starting point for a numerical approach[4].

Nonetheless we will also provide an *analytical* representation for the basis integrals in terms of scalar integrals with trivial numerators only. The price to pay in this case are inverse Gram determinants. For the calculation of a cross section this representation is numerically stable in most parts of the phase space. However, numerical problems can arise at the kinematical boundaries. In section 7 we will propose a numerical evaluation of the basis integrals which also works for exceptional kinematical configurations.

5.1 Three-point integrals

The form factors for the 3-point tensor integrals, following directly from eqs. (17) and (23), are given by

$$\begin{aligned}
A^{3,0}(S) &= I_3^n(S) \\
A_l^{3,1}(S) &= -I_3^n(l;S) \\
B^{3,2}(S) &= -\frac{1}{2}\, I_3^{n+2}(S) \\
A_{l_1 l_2}^{3,2}(S) &= I_3^n(l_1, l_2; S) \\
B_l^{3,3}(S) &= \frac{1}{2}\, I_3^{n+2}(l;S) \\
A_{l_1 l_2 l_3}^{3,3}(S) &= -I_3^n(l_1, l_2, l_3; S)
\end{aligned} \tag{64}$$

If one would like to express the integrals with nontrivial numerators above in terms of scalar integrals only, the formulae given below in eqs. (65) to (70) can be used if $\mathcal{S}$ is regular. The singular cases occur if either the 3-point function is IR divergent or one hits an anomalous threshold. Anomalous thresholds appear in scattering processes with unstable external particles. In very special kinematic situations all internal particles in a given diagram can go on-shell, which can cause integrable singularities inside the phase space [80]. One can show that in the 3-point case $\det(\mathcal{S}) = 0$ is a necessary condition for an anomalous threshold. For the case where IR divergences are present, we give a complete list of 3-point integrals with massless propagators in appendix B.

$$I_3^n(l_1;S) = \frac{b_{l_1}}{B}\left[I_3^n(S) - \sum_{j\in S} b_j\, I_2^n(S\setminus\{j\})\right] + \sum_{j\in S} \mathcal{S}^{-1}_{l_1 j} I_2^n(S\setminus\{j\}) \tag{65}$$

[4] Parameter integrals with non-trivial numerators, corresponding to tensor integrals, are also dealt with in [79], where n-dimensional tensor 3-point and 4-point functions with massless propagators are expressed by hypergeometric series expansions.

$$I_3^n(l_1,l_2;S) = -\mathcal{S}^{-1}_{l_1 l_2} I_3^{n+2}(S) + b_{l_1}(n-1) I_3^{n+2}(l_2;S) + \sum_{j\in S} \mathcal{S}^{-1}_{l_1 j}\, I_2^n(l_2;S\setminus\{j\}) \tag{66}$$

$$\begin{aligned} I_3^n(l_1,l_2,l_3;S) &= -\mathcal{S}^{-1}_{l_1 l_2} I_3^{n+2}(l_3;S) - \mathcal{S}^{-1}_{l_1 l_3} I_3^{n+2}(l_2;S) + n\, b_{l_1}\, I_3^{n+2}(l_2,l_3;S) \\ &\quad + \sum_{j\in S} \mathcal{S}^{-1}_{l_1 j}\, I_2^n(l_2,l_3;S\setminus\{j\}) \end{aligned} \tag{67}$$

$$I_3^{n+2}(S) = \frac{1}{B}\frac{1}{(n-2)}\left[I_3^n(S) - \sum_{l\in S} b_l\, I_2^n(S\setminus\{l\})\right] \tag{68}$$

$$\begin{aligned} I_3^{n+2}(l_1;S) &= \frac{1}{B}\left[b_{l_1}\, I_3^{n+2}(S) + \frac{1}{n-1}\sum_{j\in S} \mathcal{S}^{-1}_{j\, l_1}\, I_2^n(S\setminus\{j\}) \right. \\ &\quad \left. - \frac{1}{n-1}\sum_{j\in S} b_j\, I_2^n(l_1;S\setminus\{j\})\right] \end{aligned} \tag{69}$$

$$\begin{aligned} I_3^{n+2}(l_1,l_2;S) &= \frac{1}{n\,B}\left[b_{l_1}\, I_3^{n+2}(l_2;S) + b_{l_2}\, I_3^{n+2}(l_1;S) + I_3^n(l_1,l_2;S) \right. \\ &\quad \left. - \sum_{j\in S} b_j\, I_2^n(l_1,l_2;S\setminus\{j\})\right] \end{aligned} \tag{70}$$

The two-point functions I_2^n are given in appendix A. The scalar three-point function $I_3^n(S)$ is well known, see for example [46, 74, 75].

By iterating the above formulae one arrives at a representation in terms of scalar integrals with trivial numerator. As a result of the iteration, inverse Gram determinants up to the third power occur. To improve the numerical stability, the bracketing of the terms as given by eqs. (65) and (68) to (70) should be respected.

5.2 Four-point integrals

All form factors of the 4-point tensor integrals are expressed in terms of $(n+2)$- and $(n+4)$-dimensional scalar box integrals and n- and $(n+2)$-dimensional triangle integrals, with up to three Feynman parameters in the numerator.

$$A^{4,0}(S) = B\, I_4^{n+2}(S) + \sum_{j\in S} b_j\, I_3^n(S\setminus\{j\}) \tag{71}$$

$$A_l^{4,1}(S) = -b_l\, I_4^{n+2}(S) - \sum_{j\in S} \mathcal{S}^{-1}_{j\, l}\, I_3^n(S\setminus\{j\}) \tag{72}$$

$$B^{4,2}(S) = -\frac{1}{2} I_4^{n+2}(S) \tag{73}$$

$$\begin{aligned} A^{4,2}_{l_1 l_2}(S) &= b_{l_1}\, I_4^{n+2}(l_2;S) + b_{l_2}\, I_4^{n+2}(l_1;S) - \mathcal{S}^{-1}_{l_1\, l_2}\, I_4^{n+2}(S) \\ &\quad + \frac{1}{2}\sum_{j\in S}\left[\mathcal{S}^{-1}_{j\, l_2}\, I_3^n(l_1;S\setminus\{j\}) + \mathcal{S}^{-1}_{j\, l_1}\, I_3^n(l_2;S\setminus\{j\})\right] \end{aligned} \tag{74}$$

$$B_l^{4,3}(S) = \frac{1}{2} I_4^{n+2}(l;S) \tag{75}$$

$$
\begin{aligned}
A^{4,3}_{l_1 l_2 l_3}(S) &= \frac{2}{3}\,\left[\mathcal{S}^{-1}_{l_2\,l_3}\,I_4^{n+2}(l_1;S)+\mathcal{S}^{-1}_{l_1\,l_3}\,I_4^{n+2}(l_2;S)+\mathcal{S}^{-1}_{l_1\,l_2}\,I_4^{n+2}(l_3;S)\right] \\
&\quad -\left[b_{l_1}\,I_4^{n+2}(l_2,l_3;S)+b_{l_2}\,I_4^{n+2}(l_1,l_3;S)+b_{l_3}\,I_4^{n+2}(l_1,l_2;S)\right] \\
&\quad -\frac{1}{3}\sum_{j\in S}\left[\mathcal{S}^{-1}_{j\,l_1}\,I_3^n(l_2,l_3;S\setminus\{j\})+\mathcal{S}^{-1}_{j\,l_2}\,I_3^n(l_1,l_3;S\setminus\{j\})\right. \\
&\qquad\qquad \left.+\,\mathcal{S}^{-1}_{j\,l_3}\,I_3^n(l_1,l_2;S\setminus\{j\})\right] \qquad (76)
\end{aligned}
$$

$$
\begin{aligned}
C^{4,4}(S) &= \frac{1}{4}\,I_4^{n+4}(S) \qquad (77)\\
B^{4,4}_{l_1 l_2}(S) &= -\frac{1}{2}\,I_4^{n+2}(l_1,l_2;S) \qquad (78)\\
A^{4,4}_{l_1 l_2 l_3 l_4}(S) &= f^{4,4}(l_1,l_2;l_3,l_4)+f^{4,4}(l_1,l_3;l_2,l_4)+f^{4,4}(l_1,l_4;l_3,l_2) \\
&\quad +f^{4,4}(l_2,l_3;l_1,l_4)+f^{4,4}(l_2,l_4;l_3,l_1)+f^{4,4}(l_3,l_4;l_1,l_2) \\
&\quad +g^{4,4}(l_1;l_2,l_3,l_4)+g^{4,4}(l_2;l_1,l_3,l_4) \\
&\quad +g^{4,4}(l_3;l_2,l_1,l_4)+g^{4,4}(l_4;l_2,l_3,l_1)
\end{aligned}
$$

$$
\begin{aligned}
f^{4,4}(l_1,l_2;l_3,l_4) &= -\frac{1}{2}\,\mathcal{S}^{-1}_{l_1\,l_2}\,I_4^{n+2}(l_3,l_4;S) \\
g^{4,4}(l_1;l_2,l_3,l_4) &= b_{l_1}\,I_4^{n+2}(l_2,l_3,l_4;S)+\frac{1}{4}\sum_{j\in S}\mathcal{S}^{-1}_{j\,l_1}\,I_3^n(l_2,l_3,l_4;S\setminus\{j\}) \qquad (79)
\end{aligned}
$$

There are six different combinations of terms $f^{4,4}(\ldots)$ and four different combinations of terms $g^{4,4}(\ldots)$. The combinations are imposed by the symmetry of these objects and represent all different *distinguishable* index distributions. The object $f^{4,4}$ is symmetric in the first and last two indices. The symmetry in the first two indices is manifest, as $\mathcal{S}^{-1}_{l_1\,l_2}$ is symmetric. The symmetry in the last two indices is only induced when the form factors are combined with formulae (17), because the summations symmetrize these indices. The object $g^{4,4}$ is symmetric in the last three indices in the same sense.

The form factors given above are free from any inverse Gram determinant and contain only n-dimensional 3-point integrals with up to three Feynman parameters in the numerator, $(n+2)$-dimensional 3-point integrals with maximally one Feynman parameter in the numerator, and $(n+2)$- and $(n+4)$-dimensional 4-point functions with up to three respectively one Feynman parameter in the numerator. These integrals form our *basis set*, i.e. the endpoints of the reduction. We will see that for $N > 4$, no additional basis integrals are required. This basis set, being free from inverse Gram determinants, serves as an ideal starting point for a numerical evaluation.

Of course, further algebraic reduction down to integrals with no Feynman parameters in the numerator is also possible. The following formulae and the results for the 3-point integrals given above allow to achieve an algebraic representation of the form factors in terms of *scalar* integrals, i.e. integrals with no Feynman parameters in the numerator, only. Again, such a representation produces $1/B$ terms, which is equivalent to the presence of inverse Gram determinants. Exploiting the relations given in appendix D.1, the following results can be

derived:

$$I_4^{n+2}(l;S) = \frac{1}{B}\Big\{ b_l\, I_4^{n+2}(S) + \frac{1}{2}\sum_{j\in S} \mathcal{S}^{-1}_{j\,l}\, I_3^n(S\setminus\{j\}) - \frac{1}{2}\sum_{j\in S} b_j\, I_3^n(l;S\setminus\{j\})\Big\} \tag{80}$$

$$\begin{aligned} I_4^{n+2}(l_1,l_2;S) = \frac{2}{3\,B}\Big\{ & b_{l_1}\, I_4^{n+2}(l_2;S) + b_{l_2}\, I_4^{n+2}(l_1;S) \\ & -\frac{1}{2}\,\mathcal{S}^{-1}_{l_1\,l_2}\, I_4^{n+2}(S) + \frac{1}{4}\sum_{j\in S} \mathcal{S}^{-1}_{j\,l_2}\, I_3^n(l_1;S\setminus\{j\}) \\ & + \frac{1}{4}\sum_{j\in S} \mathcal{S}^{-1}_{j\,l_1}\, I_3^n(l_2;S\setminus\{j\}) - \frac{1}{2}\sum_{j\in S} b_j\, I_3^n(l_1,l_2;S\setminus\{j\})\Big\} \end{aligned} \tag{81}$$

$$\begin{aligned} I_4^{n+2}(l_1,l_2,l_3;S) = \frac{1}{2\,B}\Big\{ & b_{l_3}\, I_4^{n+2}(l_1,l_2;S) + b_{l_2}\, I_4^{n+2}(l_1,l_3;S) + b_{l_1}\, I_4^{n+2}(l_2,l_3;S) \\ & -\frac{1}{3}\Big(\mathcal{S}^{-1}_{l_1\,l_2}\, I_4^{n+2}(l_3;S) + \mathcal{S}^{-1}_{l_1\,l_3}\, I_4^{n+2}(l_2;S) + \mathcal{S}^{-1}_{l_2\,l_3}\, I_4^{n+2}(l_1;S)\Big) \\ & +\frac{1}{6}\Big(\sum_{i\in S} \mathcal{S}^{-1}_{i\,l_3}\, I_3^n(l_1,l_2;S\setminus\{i\}) + \sum_{i\in S} \mathcal{S}^{-1}_{i\,l_2}\, I_3^n(l_1,l_3;S\setminus\{i\}) \\ & + \sum_{i\in S} \mathcal{S}^{-1}_{i\,l_1}\, I_3^n(l_2,l_3;S\setminus\{i\})\Big) - \frac{1}{2}\sum_{i\in S} b_i\, I_3^n(l_1,l_2,l_3,S\setminus\{i\})\Big\} \end{aligned} \tag{82}$$

$$I_4^{n+4}(S) = \frac{1}{(n-1)\,B}\Big\{ I_4^{n+2}(S) - \sum_{j\in S} b_j\, I_3^{n+2}(S\setminus\{j\})\Big\} \tag{83}$$

$$I_4^{n+4}(l;S) = \frac{1}{n\,B}\Big\{ b_l\, I_4^{n+4}(S) + I_4^{n+2}(l;S) - \sum_{j\in S} b_j\, I_3^{n+2}(l;S\setminus\{j\})\Big\} \tag{84}$$

Applying these formulae and eq. (69), an algebraic representation of the form factors for 4-point tensor integrals in terms of 3-point functions and six-dimensional integrals without Feynman parameters in the numerator is achieved. The six-dimensional scalar box functions for massless internal lines are well known and can be found for example in [46, 71]. In section 7, the behaviour of the form factors for $B \to 0$ in this algebraic representation will be compared to the one given by eqs. (72) to (79), where inverse Gram determinants have been avoided.

6 Form factors for $N = 5$

In this section we will give numerically stable form factor representation for five point functions. We have already seen that the reduction of tensor integrals for $N \geq 6$ does not produce higher dimensional remainders, see eq. (63). The case $N = 5$ is, in a sense, the most complicated one.

The derivation is rather technical and can be found in appendix C. For five-point integrals up to rank three, the absence of higher dimensional five-point integrals already has been demonstrated by explicit calculation in [46]. In appendix C, we show that these integrals drop out for 5-point integrals of *arbitrary* rank and we derive representations which are free from both higher dimensional five-point integrals *and* inverse Gram determinants *at the same time*. To the best of our knowledge, such a representation has not been given in the literature before.

By application of eq. (C.103) and reduction of n-dimensional box integrals I_4^n to 3-point functions and higher dimensional box integrals, we obtain, dropping $\mathcal{O}(\epsilon)$ terms:

$$A^{5,0}(S) = \sum_{j\in S} b_j\, B^{\{j\}}\, I_4^{n+2}(S\setminus\{j\}) + \sum_{j\in S}\sum_{k\in S\setminus\{j\}} b_j\, b_k^{\{j\}}\, I_3^n(S\setminus\{j,k\}) \tag{85}$$

$$A_l^{5,1}(S) = -\sum_{j\in S} \mathcal{S}^{-1}_{j\,l}\, B^{\{j\}}\, I_4^{n+2}(S\setminus\{j\}) - \sum_{j\in S}\sum_{k\in S\setminus\{j\}} \mathcal{S}^{-1}_{j\,l}\, b_k^{\{j\}}\, I_3^n(S\setminus\{j,k\}) \tag{86}$$

$$B^{5,2}(S) = -\frac{1}{2}\sum_{j\in S} b_j\, I_4^{n+2}(S\setminus\{j\}) \tag{87}$$

$$\begin{aligned}A_{l_1l_2}^{5,2}(S) = {} & \sum_{j\in S}\left(\mathcal{S}^{-1}_{j\,l_1} b_{l_2} + \mathcal{S}^{-1}_{j\,l_2} b_{l_1} - 2\,\mathcal{S}^{-1}_{l_1\,l_2} b_j + b_j\, {\mathcal{S}^{\{j\}}}^{-1}{}_{l_1\,l_2}\right) I_4^{n+2}(S\setminus\{j\}) \\ & + \frac{1}{2}\sum_{j\in S}\sum_{k\in S\setminus\{j\}} \left[\mathcal{S}^{-1}_{j\,l_2}\, {\mathcal{S}^{\{j\}}}^{-1}{}_{k\,l_1} + \mathcal{S}^{-1}_{j\,l_1}\, {\mathcal{S}^{\{j\}}}^{-1}{}_{k\,l_2}\right] I_3^n(S\setminus\{j,k\})\end{aligned} \tag{88}$$

$$B_l^{5,3}(S) = \frac{1}{3}\sum_{j\in S}\left(b_j\, I_4^{n+2}(l;S\setminus\{j\}) + \frac{1}{2}\mathcal{S}^{-1}_{j\,l}\, I_4^{n+2}(S\setminus\{j\})\right) \tag{89}$$

$$\begin{aligned}A_{l_1l_2l_3}^{5,3}(S) = {} & \frac{2}{3}\sum_{j\in S}\Big[\, I_4^{n+2}(l_3;S\setminus\{j\}) \\ & \times\left(2\,\mathcal{S}^{-1}_{l_1\,l_2} b_j - \mathcal{S}^{-1}_{j\,l_1} b_{l_2} - \mathcal{S}^{-1}_{j\,l_2} b_{l_1} - b_j\, {\mathcal{S}^{\{j\}}}^{-1}{}_{l_1\,l_2}\right) + l_2\leftrightarrow l_3 + l_1\leftrightarrow l_3\Big] \\ & + \frac{1}{3}\sum_{j\in S} I_4^{n+2}(S\setminus\{j\})\, \left[\mathcal{S}^{-1}_{j\,l_3}\, {\mathcal{S}^{\{j\}}}^{-1}{}_{l_1\,l_2} + \mathcal{S}^{-1}_{j\,l_1}\, {\mathcal{S}^{\{j\}}}^{-1}{}_{l_2\,l_3} + \mathcal{S}^{-1}_{j\,l_2}\, {\mathcal{S}^{\{j\}}}^{-1}{}_{l_1\,l_3}\right] \\ & -\frac{1}{6}\sum_{j\in S}\sum_{k\in S\setminus\{j\}}\Big[\, I_3^n(l_1;S\setminus\{j,k\})\, \left(\mathcal{S}^{-1}_{j\,l_3}\, {\mathcal{S}^{\{j\}}}^{-1}{}_{k\,l_2} + \mathcal{S}^{-1}_{j\,l_2}\, {\mathcal{S}^{\{j\}}}^{-1}{}_{k\,l_3}\right) \\ & + l_1\leftrightarrow l_2 + l_1\leftrightarrow l_3\Big]\end{aligned} \tag{90}$$

$$C^{5,4}(S) = \frac{1}{4}\left(1+\frac{n-4}{3}\right)\sum_{j\in S} b_j\, I_4^{n+4}(S\setminus\{j\}) \tag{91}$$

$$\begin{aligned}B_{l_1l_2}^{5,4}(S) = {} & \frac{1}{4}\sum_{j\in S}\Bigg[\left(1+\frac{n-4}{3}\right)\left(2\,\mathcal{S}^{-1}_{l_1\,l_2} b_j - \mathcal{S}^{-1}_{j\,l_1} b_{l_2} - \mathcal{S}^{-1}_{j\,l_2} b_{l_1} - b_j\, {\mathcal{S}^{\{j\}}}^{-1}{}_{l_1\,l_2}\right) \\ & \times\, I_4^{n+4}(S\setminus\{j\}) - b_j\, I_4^{n+2}(l_1,l_2;S\setminus\{j\}) \\ & - \frac{1}{3}\mathcal{S}^{-1}_{j\,l_1}\, I_4^{n+2}(l_2;S\setminus\{j\}) - \frac{1}{3}\mathcal{S}^{-1}_{j\,l_2}\, I_4^{n+2}(l_1;S\setminus\{j\}) \\ & - \frac{1}{6}\sum_{k\in S\setminus\{j\}}\left(\mathcal{S}^{-1}_{j\,l_1}\, {\mathcal{S}^{\{j\}}}^{-1}{}_{k\,l_2} + \mathcal{S}^{-1}_{j\,l_2}\, {\mathcal{S}^{\{j\}}}^{-1}{}_{k\,l_1}\right) I_3^{n+2}(S\setminus\{j,k\})\Bigg]\end{aligned} \tag{92}$$

$$
\begin{aligned}
A^{5,4}_{l_1 l_2 l_3 l_4}(S) &= \frac{1}{4}\sum_{j\in S}\Big[f^{5,4}(l_1,l_2;l_3,l_4)+f^{5,4}(l_1,l_3;l_2,l_4)+f^{5,4}(l_1,l_4;l_2,l_3) \\
&\quad +f^{5,4}(l_2,l_3;l_1,l_4)+f^{5,4}(l_2,l_4;l_1,l_3)+f^{5,4}(l_3,l_4;l_1,l_2) \\
&\quad +g^{5,4}(l_1;l_2,l_3,l_4)+g^{5,4}(l_2;l_1,l_3,l_4) \\
&\quad +g^{5,4}(l_3;l_1,l_2,l_4)+g^{5,4}(l_4;l_1,l_2,l_3)\Big] \qquad (93)
\end{aligned}
$$

$$
\begin{aligned}
f^{5,4}(l_1,l_2;l_3,l_4) &= -2\, I_4^{n+2}(l_1,l_2;S\setminus\{j\})\,\left(2\,\mathcal{S}^{-1}_{l_3\,l_4}\,b_j-\mathcal{S}^{-1}_{j\,l_3}\,b_{l_4}-\mathcal{S}^{-1}_{j\,l_4}\,b_{l_3}-b_j\,\mathcal{S}^{\{j\}-1}{}_{l_3\,l_4}\right) \\
&\quad +\frac{1}{3}\sum_{k\in S\setminus\{j\}} I_3^{n}(l_1,l_2;S\setminus\{j,k\})\,\left(\mathcal{S}^{-1}_{j\,l_3}\,\mathcal{S}^{\{j\}-1}{}_{k\,l_4}+\mathcal{S}^{-1}_{j\,l_4}\,\mathcal{S}^{\{j\}-1}{}_{k\,l_3}\right) \\
g^{5,4}(l_1;l_2,l_3,l_4) &= -\frac{2}{3}\, I_4^{n+2}(l_1;S\setminus\{j\})\,\left(\mathcal{S}^{-1}_{j\,l_4}\,\mathcal{S}^{\{j\}-1}{}_{l_2\,l_3}+\mathcal{S}^{-1}_{j\,l_3}\,\mathcal{S}^{\{j\}-1}{}_{l_2\,l_4}+\mathcal{S}^{-1}_{j\,l_2}\,\mathcal{S}^{\{j\}-1}{}_{l_3\,l_4}\right) \\
&\qquad (94)
\end{aligned}
$$

The combinations of $f^{5,4}(\ldots)$ and $g^{5,4}(\ldots)$ appearing in $A^{5,4}_{l_1 l_2 l_3 l_4}(S)$ are imposed by the symmetry of these objects and represent all different *distinguishable* index distributions. The object $f^{5,4}$ is symmetric in the first and last two indices and the object $g^{5,4}$ is symmetric in the last three indices when combined with formulae (17).

$$
\begin{aligned}
C^{5,5}_{l}(S) &= \frac{1}{5}\sum_{j\in S}\left[-\left(1+\frac{n-4}{4}\right)\,b_j\,I_4^{n+4}(l;S\setminus\{j\})-\frac{1}{4}\,\mathcal{S}^{-1}_{j\,l}\,I_4^{n+4}(S\setminus\{j\})\right] \\
B^{5,5}_{l_1 l_2 l_3}(S) &= \frac{1}{5}\sum_{j\in S}\Bigg[\Bigg\{\frac{n}{4}\,I_4^{n+4}(l_1;S\setminus\{j\}) \\
&\quad \times\left(\mathcal{S}^{-1}_{j\,l_3}\,b_{l_2}+\mathcal{S}^{-1}_{j\,l_2}\,b_{l_3}-2\,\mathcal{S}^{-1}_{l_2\,l_3}\,b_j+b_j\,\mathcal{S}^{\{j\}-1}{}_{l_2\,l_3}\right)+l_1\leftrightarrow l_2+l_1\leftrightarrow l_3\Bigg\} \\
&\quad -\frac{1}{4}\,I_4^{n+4}(S\setminus\{j\})\,\left(\mathcal{S}^{-1}_{j\,l_3}\,\mathcal{S}^{\{j\}-1}{}_{l_1\,l_2}+\mathcal{S}^{-1}_{j\,l_2}\,\mathcal{S}^{\{j\}-1}{}_{l_1\,l_3}+\mathcal{S}^{-1}_{j\,l_1}\,\mathcal{S}^{\{j\}-1}{}_{l_2\,l_3}\right) \\
&\quad +I_4^{n+2}(l_1,l_2,l_3;S\setminus\{j\})\,b_j \\
&\quad +\left\{\frac{1}{4}\,I_4^{n+2}(l_1,l_2;S\setminus\{j\})\,\mathcal{S}^{-1}_{j\,l_3}+l_1\leftrightarrow l_3+l_2\leftrightarrow l_3\right\} \\
&\quad +\frac{1}{8}\sum_{k\in S\setminus\{j\}}\Bigg\{I_3^{n+2}(l_1;S\setminus\{j,k\}) \\
&\quad \times\left(\mathcal{S}^{-1}_{j\,l_3}\,\mathcal{S}^{\{j\}-1}{}_{k\,l_2}+\mathcal{S}^{-1}_{j\,l_2}\,\mathcal{S}^{\{j\}-1}{}_{k\,l_3}\right)+l_1\leftrightarrow l_2+l_1\leftrightarrow l_3\Bigg\}\Bigg] \qquad (95) \\
A^{5,5}_{l_1 l_2 l_3 l_4 l_5}(S) &= \frac{1}{5}\sum_{j\in S}\Big[\left(f^{5,5}(l_1,l_2,l_3;l_4,l_5)+9\text{ combinations}\right) \\
&\qquad +\left(g^{5,5}(l_1,l_2;l_3,l_4,l_5)+9\text{ combinations}\right)\Big] \qquad (96)
\end{aligned}
$$

$$f^{5,5}(l_1,l_2,l_3;l_4,l_5) = -\frac{1}{4}\sum_{k\in S\setminus\{j\}} I_3^n(l_1,l_2,l_3;S\setminus\{j,k\})\left(\mathcal{S}^{-1}_{j\,l_5}\,\mathcal{S}^{\{j\}-1}{}_{k\,l_4} + \mathcal{S}^{-1}_{j\,l_4}\,\mathcal{S}^{\{j\}-1}{}_{k\,l_5}\right)$$
$$-2\,I_4^{n+2}(l_1,l_2,l_3;S\setminus\{j\})\left(\mathcal{S}^{-1}_{j\,l_5}\,b_{l_4} + \mathcal{S}^{-1}_{j\,l_4}\,b_{l_5} - 2\,\mathcal{S}^{-1}_{l_4\,l_5}\,b_j + b_j\,\mathcal{S}^{\{j\}-1}{}_{l_4\,l_5}\right)$$

$$g^{5,5}(l_1,l_2;l_3,l_4,l_5) = \frac{1}{2}\,I_4^{n+2}(l_1,l_2;S\setminus\{j\}) \times\left(\mathcal{S}^{-1}_{j\,l_5}\,\mathcal{S}^{\{j\}-1}{}_{l_3\,l_4} + \mathcal{S}^{-1}_{j\,l_4}\,\mathcal{S}^{\{j\}-1}{}_{l_3\,l_5} + \mathcal{S}^{-1}_{j\,l_3}\,\mathcal{S}^{\{j\}-1}{}_{l_4\,l_5}\right) \qquad (97)$$

The ten different combinations of the functions $f^{5,5}(\ldots)$ and $g^{5,5}(\ldots)$ are all different distinguishable index distributions. The function $f^{5,5}$ is symmetric in the first three and last two arguments and the function $g^{5,5}$ is symmetric in the first two and last three arguments when combined with formulae (17).

7 Numerical evaluation of the basis integrals

In this section we present a method to evaluate the basis integrals of our reduction formalism numerically. The method is an alternative to an approach proposed earlier [61], which would also be viable, but which needs more analytical input. The method presented here allows to deal easily with the case of complex masses, which is necessary if unstable particles are present in the loop.

First we will explain the mathematical details of the contour deformation. Then we will show a comparison between the numerical and the algebraic implementation of some basis integrals.

7.1 Contour deformation of parameter integrals

The method which will be presented here is based on contour deformation in Feynman parameter space. As explained above, in our formalism it is sufficient to evaluate the following functions, which are the endpoints of our reduction

$$\begin{aligned}
I_3^4(j_1,\ldots,j_r) &= -\int_0^1 \prod_{i=1}^3 dz_i\,\delta(1-\sum_{l=1}^3 z_l)\frac{z_{j_1}\ldots z_{j_r}}{(-z\cdot\mathcal{S}\cdot z/2-i\delta)}\,,\\
I_3^{n+2}(j_1) &= -\Gamma(\epsilon)\int_0^1 \prod_{i=1}^3 dz_i\,\delta(1-\sum_{l=1}^3 z_l)\frac{z_{j_1}}{(-z\cdot\mathcal{S}\cdot z/2-i\delta)^{\epsilon}}\,,\\
I_4^6(j_1,\ldots,j_r) &= \int_0^1 \prod_{i=1}^4 dz_i\,\delta(1-\sum_{l=1}^4 z_l)\frac{z_{j_1}\ldots z_{j_r}}{(-z\cdot\mathcal{S}\cdot z/2-i\delta)}\,,\\
I_4^{n+4}(j_1,\ldots,j_r) &= \Gamma(\epsilon)\int_0^1 \prod_{i=1}^4 dz_i\,\delta(1-\sum_{l=1}^4 z_l)\frac{z_{j_1}\ldots z_{j_r}}{(-z\cdot\mathcal{S}\cdot z/2-i\delta)^{\epsilon}}\,, \qquad (98)
\end{aligned}$$

together with integrals of the same type, but with no Feynman parameters in the numerator, and two-point functions. We will find numerically stable representations of these integrals as

special cases of a completely general derivation which is valid for scalar integrals of the form

$$I_N^D(j_1,\ldots,j_r) \;=\; (-1)^N\Gamma(N-\frac{D}{2})\int_0^1\prod_{i=1}^N dz_i\,\delta(1-\sum_{l=1}^N z_l)\frac{z_{j_1}\ldots z_{j_r}}{(-z\cdot\mathcal{S}\cdot z/2-i\delta)^{N-D/2}} \tag{99}$$

in the case when no IR/UV divergences are present. The IR/UV singular cases are discussed below.

For loop calculations with unstable particles, it is necessary that internal propagators can be defined with complex masses, $\mathcal{M}_j^2 = M_j^2 - i\,M_j\Gamma_j$. The denominators of the integrands in eq. (99) are defined accordingly by changing the quadratic form to

$$-\frac{1}{2}z\cdot\mathcal{S}\cdot z - i\delta \rightarrow -\frac{1}{2}z\cdot\mathcal{S}\cdot z - i\left(\sum_{j=1}^N z_j\right)\left(\sum_{j=1}^N z_j M_j\Gamma_j\right) \tag{100}$$

The finite width improves the stability of a numerical integration but is not sufficient to guarantee a stable evaluation. We construct a contour deformation in those parameter integrals which belong to propagators with zero width only. This has technical reasons which will become clear below.

In a first step we eliminate the delta function in eq. (99) by decomposing the integration region into N sectors where in each sector, one of the Feynman parameters is larger than all the others.

$$1 = \sum_{l=1}^N \theta(z_l > z_1,\ldots,z_{l-1},z_{l+1},\ldots,z_N) \tag{101}$$

Note that the splitting into N sectors can be avoided by a clever choice of Feynman parametrisation adapted to the special case at hand. However, as we are interested in a general, automated approach, we do not pursue this option. The basic integral decays into N sector integrals J_l

$$I_N^D(j_1,\ldots,j_r) = (-1)^N\Gamma(N-D/2)\sum_{l=1}^N J_l(N,D,j_1,\ldots,j_r)\;. \tag{102}$$

The latter can be written as integrals over the $(N-1)$-dimensional unit hypercube. Focusing on one sector, say sector l, and introducing the vector $\vec{T} = (t_1,\ldots,t_{l-1},1,t_l,\ldots,t_{N-1})$ which defines the $N-1$ coordinates $t_1,\ldots,t_{N-1}$ of the unit hypercube, one gets:

$$J_l(N,D,j_1,\ldots,j_r) = \int_0^1\prod_{l=1}^{N-1}dt_l\,\Big(\sum_{j=1}^N T_j\Big)^{N-D-r}\frac{T_{j_1}\ldots T_{j_r}}{\Big(-T\cdot\mathcal{S}\cdot T/2-i\delta\Big)^{N-D/2}} \tag{103}$$

The denominator becomes singular if the quadratic form $Q(\vec{t}) = -T\cdot\mathcal{S}\cdot T/2$ approaches zero inside the integration region, explicitly:

$$Q(\vec{t}) = \frac{1}{2}\sum_{j,k=1}^{N-1}X_{jk}t_jt_k + \sum_{j=1}^{N-1}Y_jt_j + Z = 0\;, \tag{104}$$

where X_{jk}, Y_j and Z are defined through $\mathcal{S}_{jk}$. The singularity is protected by the $i\delta$ prescription or a finite width in some propagators. Viewing the integration volume as an $(N-1)$-dimensional hypercontour in a space with $N-1$ complex dimensions, we are looking for a contour deformation which leads to a smooth and bounded integrand without intersecting the singularity hypersurface defined by eq. (104). For the analytic continuation of $Q(\vec{t}) \to Q(\vec{x})$ we make the ansatz [36] $\vec{x} = \vec{t} - i\,\vec{\tau}$. Now:

$$Q(\vec{x}) = Q(\vec{t}) - \frac{1}{2}\sum_{j,k=1}^{N-1} X_{jk}\tau_j\tau_k - i\sum_{k=1}^{N-1}\tau_k\sum_{j=1}^{N-1}(X_{jk}t_j + Y_k) \tag{105}$$

This suggests the following choice for the deformation vector $\vec{\tau}$:

$$\begin{aligned} \vec{x}(\vec{t}) &= \vec{t} - i\,\vec{\tau}(\vec{t}) \\ \tau_k &= \begin{cases} \lambda t_k^{\alpha}(1-t_k)^{\beta}\sum\limits_{j=1}^{N-1}(X_{jk}t_j + Y_k) & \text{if } \Gamma_k = 0 \\ 0 & \text{if } \Gamma_k \neq 0 \end{cases} \end{aligned} \tag{106}$$

Note that the implementation of such a contour deformation also for the parameters which correspond to a non-vanishing width would not lead to the desired result, as $\mathrm{Im}(Q) < 0$ can not be guaranteed then. For λ, α, $\beta > 0$ the deformation moves the integration contour away from the poles, i.e. $\mathrm{Im}(Q) < 0$, without causing any harm at the boundaries.

The invariance under diffeomorphisms of the contour, $\mathcal{C}_\lambda$, means

$$\int\limits_{\mathcal{C}_0}\prod_{l=1}^{N-1} dx_l\, f(x) = \int\limits_{\mathcal{C}_\lambda}\prod_{l=1}^{N-1} dx_l\, f(x) \tag{107}$$

or in the given parametrisation:

$$\int\limits_0^1\prod_{l=1}^{N-1} dt_l\ f(\vec{t}) = \int\limits_0^1\prod_{l=1}^{N-1} dt_l\ \det\left(\frac{\partial x_i}{\partial t_j}\right)\ f(\vec{t} - i\vec{\tau}(\vec{t})) \tag{108}$$

The Jacobian is defined by

$$\frac{\partial x_l}{\partial t_j} = \delta_{lj} - i\,\lambda\, t_l^{\alpha-1}(1-t_l)^{\beta-1}\Big[\delta_{lj}[\alpha(1-t_l) - \beta t_l]\Big(\sum_{k=1}^{N-1} X_{lk}t_k + Y_l\Big) + t_l(1-t_l)X_{lj}\Big]\bar{\delta}(\Gamma_l)$$

Here $\bar{\delta}(\Gamma_l)$ is equal to one if $\Gamma_l = 0$ and equal to zero else. To prove the invariance one can closely follow the derivation presented in the appendix of [36], apart from the fact that in our case a surface term is present which however turns out to be zero for the proposed contour deformation (106).

Some comments are in order. To simplify the discussion let us assume that $\Gamma_l = 0$ for all l.

- While $\lambda \nabla \cdot Q$ controls the size of the deformation, α, β control the smoothness of the deformation at the integration boundaries. The vanishing of the gradient of $Q(\vec{x})$ *inside* the integration volume is only critical if $Q \to 0$ at the same time. Using $\tilde{X}_{ij}^{-1} = X_{ij}^{-1} \det(X)$, this critical situation is defined by the equations

$$\det(X)\; Z = \frac{1}{2} \sum_{l,j=1}^{N-1} \tilde{X}_{lj}^{-1} Y_l Y_j$$
$$0 < -\mathrm{sgn}(\det(X)) \sum_{l=1}^{N-1} \tilde{X}_{lj}^{-1} Y_l < |\det(X)| \quad , \quad j \in \{1, \ldots, N-1\} \tag{109}$$

which are polynomial in the kinematical invariants. This situation corresponds to an anomalous threshold which is an exceptional kinematical configuration. With respect to integration over the phase space of external particles this is an integrable singularity. The critical surface in the integration regions is given by

$$t_j \;=\; t_j^{\mathrm{crit.}} = -\sum_{l=1}^{N-1} X_{jl}^{-1} Y_l \tag{110}$$

All one has to do is to split the integration hypercube subsequently into 2^{N-1} parts. This maps the critical surface to the integration boundary where adaptive integration routines can cope with the problem. If $t_j^{\mathrm{crit.}} = 0$ or 1 subleading singularities may be probed.

- In the case of an UV divergent integral it is easy to explicitly isolate the UV pole. At one loop, only overall UV divergences are present, which manifest themselves in terms of Γ-functions in front of the parameter integral. With $D = 4 + 2m - 2\epsilon$ one has an UV divergence, if and only if $N \leq 2 + m$. Then $\Gamma(N - D/2)$ has a single pole in ϵ. In this case we need the sector integral J_l to order ϵ:

$$J_l(N, D, j_1, \ldots, j_r) = \int_0^1 \prod_{l=1}^{N-1} dt_l \left(\sum_{j=1}^{N} T_j\right)^{N-4-2m-r} \left(-\frac{1}{2} T \cdot \mathcal{S} \cdot T - i\delta\right)^{2+m-N}$$
$$\times \left(T_{j_1} \ldots T_{j_r}\right) \left[1 - \epsilon \log\left(-\frac{1}{2} T \cdot \mathcal{S} \cdot T - i\delta\right) + 2\epsilon \log\left(\sum_{j=1}^{N} T_j\right) + \mathcal{O}(\epsilon^2)\right] \tag{111}$$

If $N < 2+m$, this integral is numerically stable without any modifications. The $\mathcal{O}(1)$ term is a real number multiplying the UV pole. If $N = 2+m$, there is a logarithmic singularity in the $\mathcal{O}(\epsilon)$ term which can be dealt with by using the same contour deformation as defined in eq. (106). Note that no IR poles are present in the case $N \leq 2 + m$ (the UV case). The UV and IR problems are thus nicely separated.

- The case of IR divergent 3-point functions can be treated analytically[5]. We give a complete list of 3-point functions for internal masses equal to zero in appendix B.

[5]It is also possible to extract the infrared poles by the method of iterated sector decomposition [82] from the parameter representations. This can be done in a completely automated way. A Laurent series in ϵ is produced, where the coefficients are again Feynman parameter integrals with slightly more complicated denominators. The contour deformation can then be applied to a reduced number of Feynman parameters.

We note that the presented method should also be applicable for the direct numerical computation of finite Feynman parameter integrals without doing any algebraic tensor reduction. This is presently under investigation.

7.2 Comparison between numerical and algebraic approach

We present now a comparison between the algebraical and the numerical approach of evaluating the basis functions of our reduction formalism.

As explained above we have algebraic representations of the higher dimensional 4-point functions where inverse Gram determinants are present, see section 5. In a realistic application one expects that these representations will be numerically well-behaved in the bulk of the phase space under consideration. However, when approaching exceptional kinematical situations, compensations of large numbers will happen which will finally spoil a reliable evaluation.

On the other hand, one expects that the method where integrals with Feynman parameters in the numerator are evaluated numerically from the start should not be sensitive to the presence of inverse Gram determinants, as there are no singular denominators in this representation. We have implemented the method for $(n+2)$-dimensional box and n-dimensional triangle functions for general kinematics and obtain good numerical behaviour using standard deterministic and Monte-Carlo methods. The two- and three-dimensional integral representations allow for a reliable direct evaluation of the required integrals. Comparing our numerical implementation to the algebraic one, we found that the algebraic implementation is much faster and accurate in the interior of the phase space, while the numerical one allows for the automated evaluation of the integrals near exceptional momentum configurations. This will be illustrated in the following by a simple example.

Let us consider a $2 \to 2$ process $p_1 + p_2 \to p_3 + p_4$ with $p_1^2 = p_2^2 = M^2$, where the momenta are parametrised as

$$\begin{aligned} p_1 &= (E(x), 0, 0, M\,x)\,, & p_2 &= (E(x), 0, 0, -M\,x) \\ p_3 &= E(x)\,(1, 0, \sin\theta, \cos\theta), & p_4 &= E(x)\,(1, 0, -\sin\theta, -\cos\theta) \\ E(x) &= M\,\sqrt{1+x^2} \end{aligned}$$

The Gram determinant is given by $\det(G) = 32M^6\,(1+x^2)^2\,x^2\,\sin^2\theta$. Exceptional configurations are the forward/backward scattering region, $\theta = 0, \pi$ and the case $x = 0$.

In Fig. 7.2 we plot the real and imaginary parts of the functions $I_4^6(1)$, $I_4^6(z_4)$ with the parameter x varied from 1 to 0. For the plots we have set $M = 1$ and $\theta = 7\pi/30$. The output for the plot was obtained by a Fortran code working in double precision. Whereas the 6-dimensional box function with numerator equal to one, $I_4^6(1)$, shows a perfect agreement with the result from the algebraic representation for x values as low as $x \sim 10^{-6}$, the box with a numerator, $I_4^6(z_4)$, already starts to fluctuate severely for values of x as large as $x \sim 10^{-3}$. However, for both of these integrals the numerical instabilities in the algebraic implementation occur in a region where the result could be safely extrapolated to the boundary $x = 0$ for our kinematical situation. In Fig. 7.2 the same plots are shown for the functions $I_4^6(z_3 z_4)$ and $I_4^6(z_3 z_4^2)$. The algebraic representations of these integrals have higher powers of inverse Gram determinants and are thus less stable. It is important to note that the instability occurs already

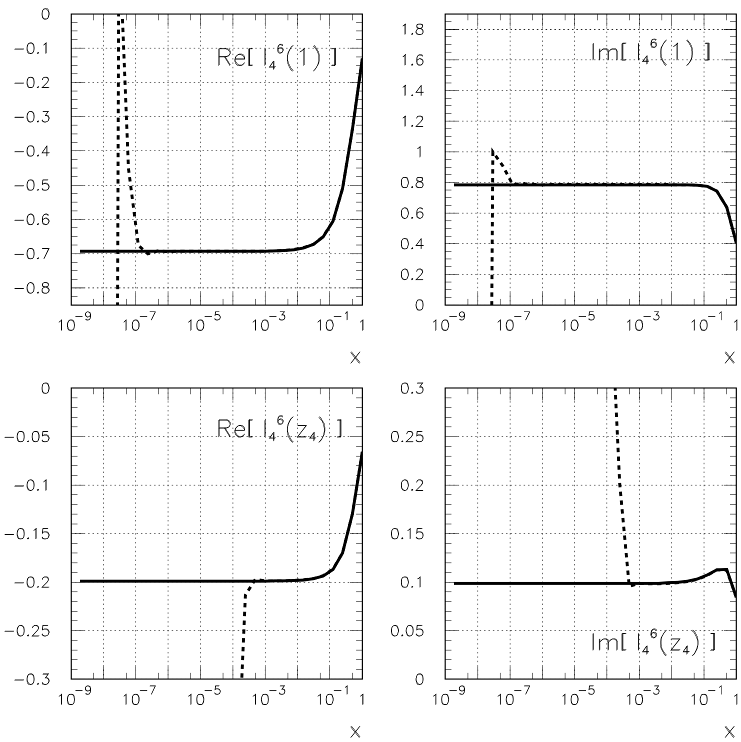

Figure 3: Real and imaginary parts of the basis integrals $I_4^6(1)$ and $I_4^6(z_4)$, plotted versus the parameter x which interpolates between exceptional and non-exceptional kinematics, as explained in the text. The solid line stems from the numerical implementation, the dashed curves show the numerical behaviour of the algebraic representation.

for values of x where one cannot yet safely extrapolate to the integration boundary. We have checked that applying quadruple precision for the evaluation of the discussed integrals leads to an improved behaviour. The function $I_4^6(1)$ can be evaluated correctly for the whole plotted x-range for the given kinematics. The functions $I_4^6(z_4)$, $I_4^6(z_3 z_4)$ and $I_4^6(z_3 z_4^2)$ show instabilities below $x \sim 10^{-7}$, $x \sim 10^{-6}$ and $x \sim 10^{-4}$ respectively. The numerical problems are confined to a smaller phase space region but are still present. Of course the analytical approach could be improved by an algebraic expansion of the expression around critical regions, as has been done for example in [47, 55]. Methods relying on such a Taylor expansion are faster, but require additional manual work, whereas the method suggested here is automated.

Note that in all cases the purely numerical implementation is completely stable. It is actually possible to evaluate the given integrals numerically for all degenerate cases $x = 0$ and $\theta = 0, \pi$.

The evaluation time for each plot point with an accuracy of better than one per cent using the numerical method is of the order of seconds on a standard PC with a Pentium 4 processor if Monte Carlo methods are applied. A precision of one per cent for the higher order correction should be well sufficient for phenomenological applications. The analytical evaluation is of the order of milliseconds and precise to standard Fortran double precision. Given the fact that the

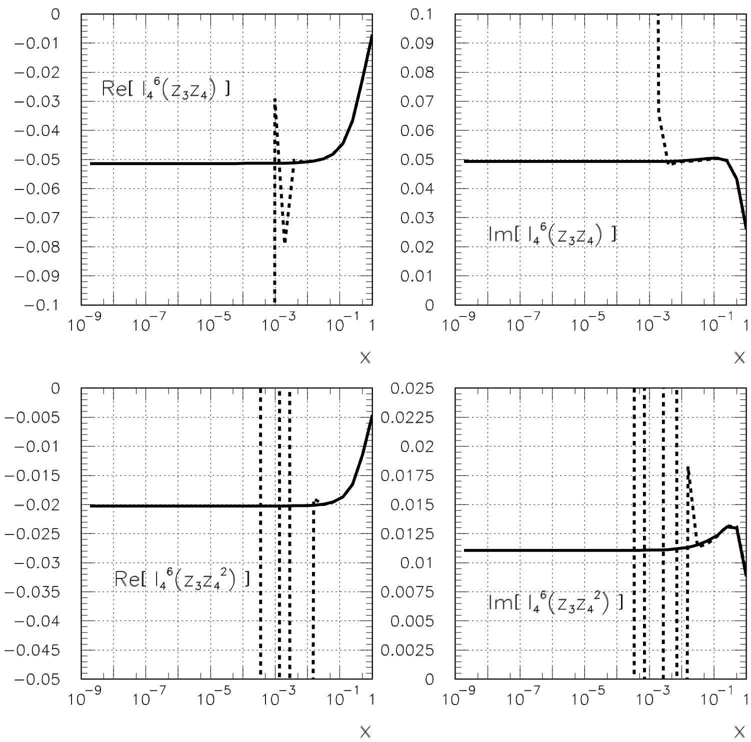

Figure 4: Same as Fig. 7.2 but for the basis integrals $I_4^6(z_3 z_4)$ and $I_4^6(z_3 z_4^2)$.

numerical evaluation is only called in a very small fraction of the phase space the relative speed is compensated by the small size of the critical regions.

We conclude from this study that the best procedure for practical applications is to use numerical implementations near exceptional kinematical configurations and analytical ones in the interior phase space domains.

8 Recipe for the practitioner

In this section we would like to summarise how to apply the main results of the article in a practical calculation. Two steps have to be distinguished: first, expressing an amplitude in terms of our basis integrals and second, the evaluation of the basis integrals.

Expressing the amplitude in terms of basis functions

After having generated the amplitude as a combination of Feynman diagrams, it has to be expressed in terms of tensor integrals as defined in eq. (14). The further processing of these tensor integrals depends on the number N of external legs.

In Fig. 5 we show the decision tree which indicates where to find the formulae to reduce an

N-point tensor integral to our basis integrals. The latter are 2-point functions, n- and $(n+2)$-dimensional 3-point functions and $(n+2)$- and $(n+4)$-dimensional 4-point functions with up to three Feynman parameters in the numerator. To perform the reduction with algebraic programs it is sufficient to look up and code the given equation numbers. Up to this point the representation is free from inverse Gram determinants. Algebraic simplifications might be applied after reduction for the coefficients of a given basis integral before proceeding to the numerical evaluation of the amplitude. We give a list of useful relations in appendices D and E.

As an illustration for a tensor reduction to basis integrals, let us show two examples. The first is the explicit expression for a rank two 5-point integral:

$$\begin{aligned}
I_5^{n,\mu_1\mu_2}(a_1,a_2;S) &= \int d\bar{k}\, \frac{q_{a_1}^{\mu_1}\, q_{a_2}^{\mu_2}}{\prod_{i\in S}(q_i^2-m_i^2+i\delta)} \\
&= g^{\mu_1\mu_2}\, B^{5,2}(S) + \sum_{l_1,l_2\in S} \Delta_{l_1\,a_1}^{\mu_1}\, \Delta_{l_2\,a_2}^{\mu_2}\, A_{l_1\,l_2}^{5,2}(S) \\
B^{5,2}(S) &= -\frac{1}{2}\sum_{j\in S} b_j\, I_4^{n+2}(S\setminus\{j\}) \\
A_{l_1 l_2}^{5,2}(S) &= \sum_{j\in S} \left(\mathcal{S}^{-1}_{j\,l_1}\, b_{l_2} + \mathcal{S}^{-1}_{j\,l_2}\, b_{l_1} - 2\,\mathcal{S}^{-1}_{l_1\,l_2}\, b_j + b_j\, \mathcal{S}^{\{j\}-1}{}_{l_1\,l_2}\right) I_4^{n+2}(S\setminus\{j\}) \\
&\quad + \frac{1}{2}\sum_{j\in S}\sum_{k\in S\setminus\{j\}} \left[\mathcal{S}^{-1}_{j\,l_2}\, \mathcal{S}^{\{j\}-1}{}_{k\,l_1} + \mathcal{S}^{-1}_{j\,l_1}\, \mathcal{S}^{\{j\}-1}{}_{k\,l_2}\right] I_3^n(S\setminus\{j,k\})\,.
\end{aligned}$$

As one can see, there is no inverse Gram determinant, only the inverse of the kinematic matrix $\mathcal{S}$ is present[6].

A rank one 6-point integral has the form

$$I_6^{n,\mu}(a;S) = -\sum_{j\in S} \mathcal{C}^{\mu}_{j\,a}\, I_5^n(S\setminus\{j\}) \tag{112}$$

$$= -\sum_{j,l\in S} \Delta^{\mu}_{la}\, \mathcal{S}^{-1}_{lj} \sum_{k\in S} b_k^{\{j\}} \left[B^{\{j,k\}}\, I_4^{n+2}(S\setminus\{j,k\}) + \sum_{m\in S\setminus\{j,k\}} b_m^{\{j,k\}}\, I_3^n(S\setminus\{j,k,m\}) \right]\,.$$

In both examples the basis integrals are already scalar integrals without Feynman parameters in the numerator. For higher rank tensor integrals this is not the case anymore.

Evaluation of the basis integrals

The case $N=2$ needs no extra discussion. All necessary formulae are gathered in appendix A. For the evaluation of the remaining basis integrals we distinguish two cases, depending

[6] The determinant $\det\mathcal{S}$ is vanishing only if one hits an anomalous threshold [80, 81], which corresponds to an integrable singularity. The location of anomalous thresholds depends on the external kinematics and the internal particle masses. Note that there are no anomalous thresholds in the physical region if all internal masses are zero. This is also true for the massive case as long as the external particles are stable.

on whether the Gram determinants are numerically problematic or not. By eq. (45) Gram determinants are related to the quantity B, the sum of reduction coefficients. Technically, the splitting into safe and problematic regions can be achieved by introducing an adequate energy scale Λ. This scale should be chosen such that for $B\Lambda^2 \geq 1$ the evaluation of the *analytical* expression for the basis integrals in terms of purely scalar integrals is numerically stable. In this case the evaluation of the basis integrals for $N = 3$ can be done by analytic reduction to scalar integrals by using eqs. (68)-(70) and eqs. (A.1)-(A.10). If the 3-point functions are IR divergent, reduction formulae are not applicable as $\det \mathcal{S} = 0$. In this case one has to use appendix B, where explicit representations for all 3-point functions with and without Feynman parameters in the numerator for massless internal particles are provided. We do not quote the formulae for the IR divergent integrals where internal masses are present. Note that *all* 3-point functions with IR poles are relatively simple functions which can always be treated analytically.

We illustrate our evaluation strategy for the 3-point functions in Fig. 6, where we indicate all relevant sections or equation numbers needed for the implementation of our method.

The evaluation strategy for the $N = 4$ basis integrals is depicted in Fig. 7. There are no IR divergences in this case. Note that the analytic branch in Fig. 7 contains implicitly evaluations of 3-point functions given in fig. 6 and an evaluation of scalar box integrals in 6-dimensions. For massless propagators analytical representations can be found in [46, 71]. To our best knowledge no complete list of these integrals for all combinations of internal and external masses is available in the literature. However, we emphasise that the numerical evaluation, using the contour deformation method introduced in section 7.1, can always be used if the analytical representation is not known. Analytical and numerical representation are complementary to each other in the sense that the former are fast and accurate in the main part of the phase space. The slower but robust numerical implementation also works well for exceptional kinematical configurations. As these dangerous phase space regions only cover a small part of the phase space, the speed issue does not pose a problem.

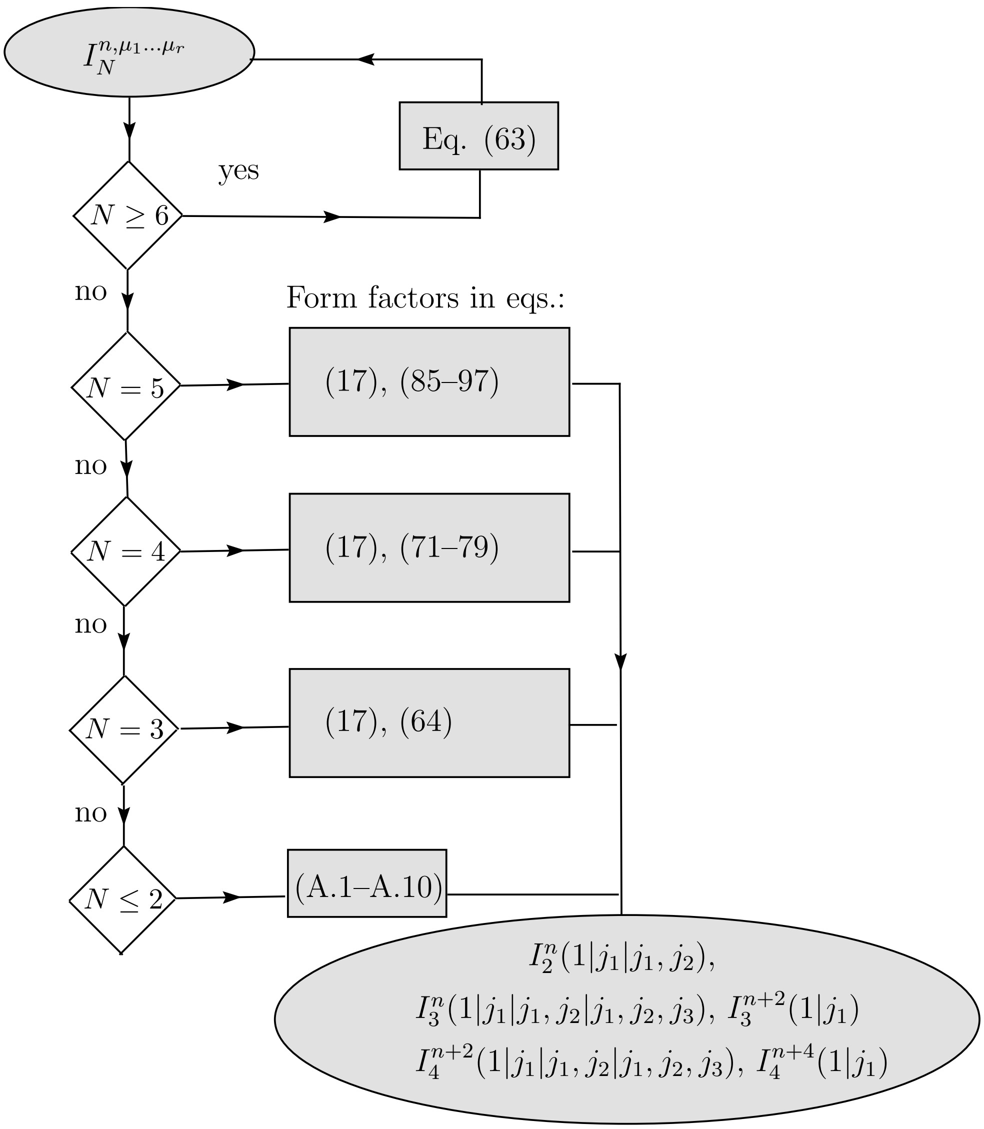

Figure 5: Reduction of N-point tensor integrals to basis integrals. The indicated equations can be implemented directly into an algebraic computer program. $I_N^n(1|j_1|j_1,j_2|j_1,j_2,j_3)$ denotes the integral I_N^n with zero, one, two or three Feynman parameters in the numerator.

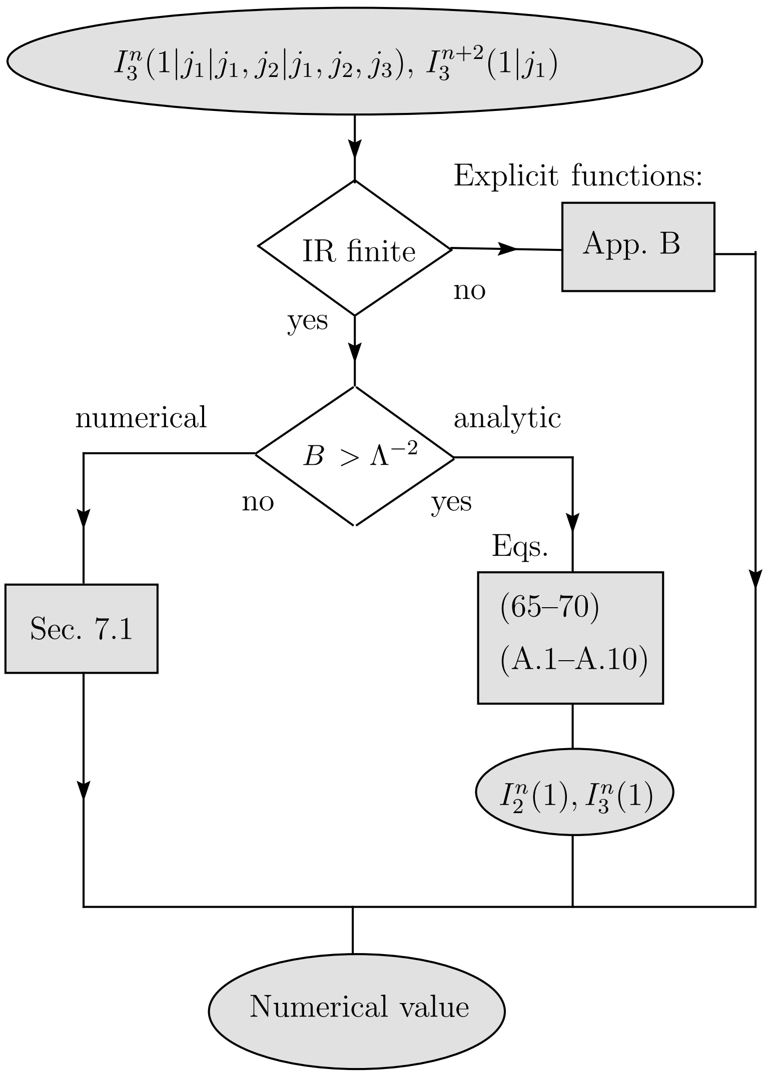

Figure 6: Evaluation of the basis integrals: the triangle case, $N = 3$.

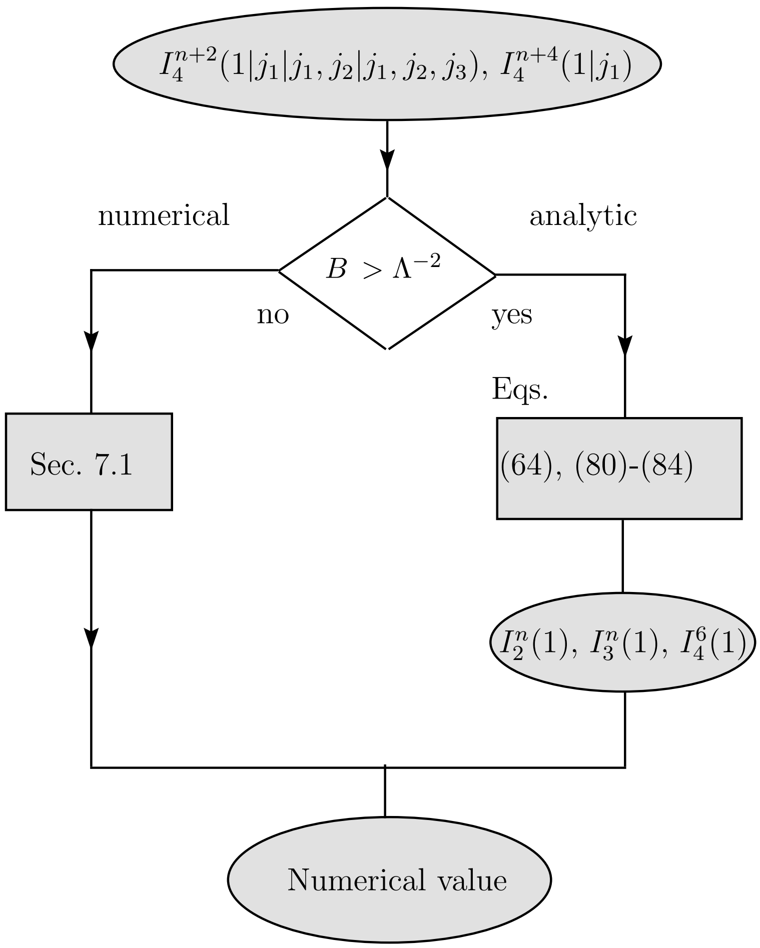

Figure 7: Evaluation of the basis integrals: the box case, $N = 4$. For the analytical evaluation of eqs. (64) use Fig. 6.

9 Conclusion

In this paper, we have presented a complete method for the calculation of one-loop multi-leg amplitudes. In principle it can be applied to arbitrary N-point problems, the limitation coming only from computer power. We offer a new method for tensor reduction which has several advantages:

- The formalism, using dimensional regularisation, is valid for both massless and massive particles.
- Infrared divergences are easily isolated by construction. They appear only in terms of three-point integrals which we list explicitly for the massless case.
- The formalism is completely shift invariant, such that its iterated application does not require the redefinition of loop momenta.
- Integrals in more than $n = 4 - 2\epsilon$ dimensions do not have to be evaluated for $N \geq 5$ external legs, as we have proven that they drop out.
- By using Feynman parameter integrals with non-trivial numerators as basis functions, inverse Gram determinants can be completely avoided.

We present two methods to compute the basis integrals which are the endpoints of our reduction. First we discuss the possibility of a "purely algebraic" approach, where all non-scalar integrals are reduced further to end up with scalar integrals only. This procedure re-introduces inverse Gram determinants, which could spoil the subsequent numerical evaluation if an exceptional kinematic configuration is approached. We also propose a method to compute the basis integrals completely numerically by multi-dimensional contour deformation in Feynman parameter space.

We have compared the two alternatives, with special emphasis on the behaviour for exceptional kinematics. We show that the semi-numerical approach, where non-scalar integrals are used as basis integrals, is very stable if an exceptional kinematic configuration is approached. On the other hand, the evaluation of the same integrals in a representation where they have been reduced algebraically down to scalar integrals, is stable only in the interior of the phase space. In this region however – which is the bulk of the phase space – their evaluation is of course faster than evaluating the non-scalar form. In a program to calculate one-loop amplitudes, it is possible to combine the virtues of both alternatives by using the algebraic representation in the interior phase space domains and switching to the semi-numerical one at the phase space boundaries.

The paper contains a complete list of form factors for integrals of rank $r \leq N$, where $N = 1, \ldots, 5$, and it is shown that for $N > 5$ it is not necessary to introduce new form factors. The formalism naturally maps tensor N-point integrals with $N \geq 6$ to combinations of 5-point integrals and reduction coefficients.

All the formulae needed for a direct implementation of the formalism are given in the paper, except some well-known 3-point and 4-point integrals, such that readers who are less interested

in the technical details can straightforwardly use the method by following the guidelines in the section “recipe for the practitioner”.

Further, we list several relations, between form factors as well as between reduction coefficients, which can be very useful to achieve a more compact form of a given amplitude and/or to perform checks of the program. In particular, we give a helicity decomposition for the case of massless 6-point functions which leads to very compact expressions for the reduction coefficients.

In summary, the formalism presented here, as it can deal with massive particles as well as infrared divergences, has no restriction on the number of external legs and is suitable for numerical integration, can be used as the basis of a general program to calculate multi-leg one-loop amplitudes efficiently in a highly automated way.

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A Form factors for $N = 1, 2$

For completeness, we provide the one- and two-point functions in this appendix. We give the kinematical arguments here in terms of invariants, as $\mathcal{S}$ is trivial in these cases.

$$I_1^{n,\mu_1}(a_1; m_1^2) \;=\; \Delta^{\mu_1}_{1\,a_1} I_1^n(m_1^2) = \Delta^{\mu_1}_{1\,a_1}\, m_1^2\, I_2^n(0,0,m_1^2) \tag{A.1}$$

Here we use the fact that 1-point functions may be written as degenerate 2-point functions.

The kinematical matrix for the 2-point functions is

$$\mathcal{S} = -\begin{pmatrix} 2\,m_1^2 & -s+m_1^2+m_2^2 \\ -s+m_1^2+m_2^2 & 2\,m_2^2 \end{pmatrix} \tag{A.2}$$

The Lorentz tensor decomposition for the tensor 2-point functions

$$I_2^{n\,,\mu_1}(a_1; s, m_1^2, m_2^2) \;=\; \sum_{l_1 \in S} \Delta^{\mu_1}_{l_1\,a_1}\; A^{2,1}_{l_1}(S) \tag{A.3}$$

$$I_2^{n,\mu_1\mu_2}(a_1,a_2;s,m_1^2,m_2^2) \;=\; g^{\mu_1\,\mu_2}\,B^{2,2}(S) + \sum_{l_1,l_2\in S} \Delta^{\mu_1}_{l_1\,a_1}\,\Delta^{\mu_2}_{l_2\,a_2}\,A^{2,2}_{l_1\,l_2}(S) \tag{A.4}$$

defines the form factors

$$\begin{aligned}
A_1^{2,1}(s,m_1^2,m_2^2) &= -\frac{1}{2}I_2^n(s,m_1^2,m_2^2) \\
&\quad +\frac{m_1^2-m_2^2}{2\,s}[I_2^n(s,m_1^2,m_2^2)-I_2^n(0,m_1^2,m_2^2)] \\
B^{2,2}(s,m_1^2,m_2^2) &= \frac{1}{2(n-1)}\Big[2\,m_2^2 I_2^n(s,m_1^2,m_2^2)+m_1^2 I_2^n(0,0,m_1^2) \\
&\quad +\frac{-s+m_1^2-m_2^2}{2}\Big(I_2^n(s,m_1^2,m_2^2) \\
&\quad -\frac{m_1^2-m_2^2}{s}[I_2^n(s,m_1^2,m_2^2)-I_2^n(0,m_1^2,m_2^2)]\Big)\Big] \\
A_{11}^{2,2}(s,m_1^2,m_2^2) &= \frac{1}{2(n-1)s}\Big[\frac{n(s-m_1^2+m_2^2)}{2}\Big(I_2^n(s,m_1^2,m_2^2) \\
&\quad -\frac{m_1^2-m_2^2}{s}[I_2^n(s,m_1^2,m_2^2)-I_2^n(0,m_1^2,m_2^2)]\Big) \\
&\quad -2\,m_2^2\,I_2^n(s,m_1^2,m_2^2)+\,(n-2)\,m_1^2 I_2^n(0,0,m_1^2)\Big]
\end{aligned} \tag{A.5}$$

The other tensor coefficients are obtained by the following relations

$$\begin{aligned}
A_1^{2,1}+A_2^{2,1} &= -I_2^n \\
A_{11}^{2,2}+A_{12}^{2,2} &= -A_1^{2,1} \\
A_{21}^{2,2}+A_{22}^{2,2} &= -A_2^{2,1} \\
A_{11}^{2,2}+A_{12}^{2,2}+A_{21}^{2,2}+A_{22}^{2,2} &= I_2^n
\end{aligned} \tag{A.6}$$

which follow directly from the fact that in one-loop parameter integrals, the sum of all Feynman parameters is equal to one. The general 2-point scalar integral is well known [75]. We give here the integral representation for completeness.

$$I_2^n(s,m_1,m_2) \;=\; \Gamma(\epsilon) - \int_0^1 dx\,\log(-s\,x\,(1-x)+xm_1^2+(1-x)m_2^2-i\delta)+\mathcal{O}(\epsilon) \tag{A.7}$$

If the external vector is light-like the formulae degenerate to

$$\begin{aligned}
A_1^{2,1}(0,m_1^2,m_2^2) &= +\frac{(4\,m_1^2-n\,m_1^2+n\,m_2^2)\,m_1^2}{2n(m_2^2-m_1^2)^2}\,I_2^n(0,0,m_1^2) \\
&\quad +\frac{(-4\,m_2^2-n\,m_1^2+n\,m_2^2)\,m_2^2}{2n(m_2^2-m_1^2)^2}\,I_2^n(0,0,m_2^2) \\
&\quad -\frac{1}{2(m_1^2-m_2^2)}\,(m_1^2\,I_2^n(0,0,m_1^2)-m_2^2\,I_2^n(0,0,m_2^2)) \\
B^{2,2}(0,m_1^2,m_2^2) &= -\frac{m_1^2}{n(m_2^2-m_1^2)}\,I_2^n(0,0,m_1^2)+\frac{m_2^2}{n(m_2^2-m_1^2)}\,I_2^n(0,0,m_2^2)
\end{aligned}$$

$$
\begin{aligned}
A_{11}^{2,2}(0,m_1^2,m_2^2) &= \frac{1}{(n+2)n(m_1^2-m_2^2)^3}\Big([(m_1^2-m_2^2)^2\,n\,(n+2) \\
&\quad -4\,m_1^2\,(n\,(m_1^2-m_2^2)-2\,m_2^2)]\,m_1^2\,I_2^n(0,0,m_1^2) \\
&\quad -8\,m_2^4\,I_2^n(0,0,m_2^2)\Big)
\end{aligned} \tag{A.8}
$$

where

$$
\begin{aligned}
I_2^n(0,m_1^2,m_2^2) &= \frac{m_2^2\,I_2^n(0,0,m_2^2)-m_1^2\,I_2^n(0,0,m_1^2)}{m_2^2-m_1^2} \\
I_2^n(0,0,m_1^2) &= \frac{I_1^n(m_1^2)}{m_1^2} = \frac{2}{n-2}I_2^n(0,m_1^2,m_1^2) = -\Gamma(1-n/2)\,(m_1^2)^{n/2-2}
\end{aligned} \tag{A.9}
$$

In the case $m_1 = m_2$ one finds

$$
\begin{aligned}
A_1^{2,1}(0,m_1^2,m_1^2) &= \frac{n-2}{4}\,I_2^n(0,0,m_1^2) \\
B^{2,2}(0,m_1^2,m_1^2) &= \frac{m_1^2}{2}\,I_2^n(0,0,m_1^2) \\
A_{11}^{2,2}(0,m_1^2,m_1^2) &= \frac{n-2}{6}\,I_2^n(0,0,m_1^2)
\end{aligned} \tag{A.10}
$$

Two-point functions with no scale at all are defined as zero in dimensional regularisation.

B Divergent three–point functions with massless propagators

We consider here only three-point functions with massless propagators. The parameter representation is given by

$$
\begin{aligned}
I_3^n(j_1,j_2,j_3;S) &= -\Gamma(3-\frac{n}{2})\int_0^1\prod_{i=1}^3 dz_i\,\delta(1-\sum_{l=1}^3 z_l)\,z_{j_1}z_{j_2}z_{j_3}\,(R^2)^{\frac{n}{2}-3} \\
R^2 &= -z_1\,z_2\,\mathcal{S}_{12}-z_2\,z_3\,\mathcal{S}_{23}-z_1\,z_3\,\mathcal{S}_{13}\,-i\,\delta
\end{aligned} \tag{B.11}
$$

If one or two invariants out of the set $\{\mathcal{S}_{12},\mathcal{S}_{23},\mathcal{S}_{13}\}$ vanish, one gets an IR divergence. In this case one has $\det\mathcal{S}=0$, such that the formulae given in section 5.1 do not apply. Therefore we provide analytic representations for all 3-point parameter integrals with massless propagators here. An overall coefficient r_Γ is defined as

$$
r_\Gamma = \frac{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)}{\Gamma(1-2\,\epsilon)}\,.
$$

For three-point functions with one non-zero invariant, denoted by X, we labelled the internal propagators in such way that $\mathcal{S}_{13}=X$ and $\mathcal{S}_{12}=\mathcal{S}_{23}=0$. For three-point functions with two non-zero invariants X and Y, we set $\mathcal{S}_{23}=X$ and $\mathcal{S}_{13}=Y$. Thus the integrals $I_3^n(z_i,z_j,\ldots;0,0,X)$ are symmetric under exchange of $z_1\leftrightarrow z_3$ and the $I_3^n(z_i,z_j,\ldots;0,X,Y)$

are symmetric under simultaneous exchange of $z_1 \leftrightarrow z_2$ and $X \leftrightarrow Y$. We obtain for integrals with no Feynman parameters in the numerator:

$$I_3^n(0,0,X) = \frac{r_\Gamma}{\epsilon^2} H_0(X,-\epsilon) \tag{B.12}$$

$$I_3^n(0,X,Y) = \frac{r_\Gamma}{\epsilon^2} H_1(X,Y,-\epsilon) \tag{B.13}$$

with one Feynman parameter:

$$I_3^n(1;0,0,X) = -\frac{r_\Gamma}{\epsilon} \frac{1}{1-2\,\epsilon} H_0(X,-\epsilon) \tag{B.14}$$

$$I_3^n(2;0,0,X) = \frac{r_\Gamma}{\epsilon^2} \frac{1}{1-2\,\epsilon} H_0(X,-\epsilon) \tag{B.15}$$

$$I_3^n(3;0,0,X) = -\frac{r_\Gamma}{\epsilon} \frac{1}{1-2\,\epsilon} H_0(X,-\epsilon) \tag{B.16}$$

$$I_3^n(1;0,X,Y) = \frac{r_\Gamma}{\epsilon^2} \frac{1-\epsilon}{1-2\,\epsilon} H_2(X,Y,-\epsilon) \tag{B.17}$$

$$I_3^n(2;0,X,Y) = \frac{r_\Gamma}{\epsilon^2} \frac{1-\epsilon}{1-2\,\epsilon} H_2(Y,X,-\epsilon) \tag{B.18}$$

$$I_3^n(3;0,X,Y) = -\frac{r_\Gamma}{\epsilon} \frac{1}{1-2\,\epsilon} H_1(X,Y,-\epsilon) \tag{B.19}$$

with two Feynman parameters:

$$I_3^n(1,1;0,0,X) = -\frac{r_\Gamma}{\epsilon} \frac{1}{2(1-2\epsilon)} H_0(X,-\epsilon) \tag{B.20}$$

$$I_3^n(2,2;0,0,X) = \frac{r_\Gamma}{\epsilon^2} \frac{1}{(1-\epsilon)(1-2\epsilon)} H_0(X,-\epsilon) \tag{B.21}$$

$$I_3^n(3,3;0,0,X) = -\frac{r_\Gamma}{\epsilon} \frac{1}{2(1-2\epsilon)} H_0(X,-\epsilon) \tag{B.22}$$

$$I_3^n(1,2;0,0,X) = -\frac{r_\Gamma}{\epsilon} \frac{1}{2(1-\epsilon)(1-2\epsilon)} H_0(X,-\epsilon) \tag{B.23}$$

$$I_3^n(1,3;0,0,X) = \frac{r_\Gamma}{2(1-\epsilon)(1-2\epsilon)} H_0(X,-\epsilon) \tag{B.24}$$

$$I_3^n(2,3;0,0,X) = -\frac{r_\Gamma}{\epsilon} \frac{1}{2(1-\epsilon)(1-2\epsilon)} H_0(X,-\epsilon) \tag{B.25}$$

$$I_3^n(1,1;0,X,Y) = \frac{r_\Gamma}{\epsilon^2} \frac{2-\epsilon}{2(1-2\epsilon)} H_3(X,Y,-\epsilon) \tag{B.26}$$

$$I_3^n(2,2;0,X,Y) = \frac{r_\Gamma}{\epsilon^2} \frac{2-\epsilon}{2(1-2\epsilon)} H_3(Y,X,-\epsilon) \tag{B.27}$$

$$I_3^n(3,3;0,X,Y) = -\frac{r_\Gamma}{\epsilon} \frac{1}{2(1-2\epsilon)} H_1(X,Y,-\epsilon) \tag{B.28}$$

$$I_3^n(1,2;0,X,Y) = \frac{r_\Gamma}{\epsilon^2} \frac{2-\epsilon}{2(1-2\epsilon)} \left(H_2(X,Y,-\epsilon) - H_3(X,Y,-\epsilon)\right) \tag{B.29}$$

$$I_3^n(1,3;0,X,Y) = -\frac{r_\Gamma}{\epsilon}\,\frac{1}{2(1-2\epsilon)}\,H_2(X,Y,-\epsilon) \tag{B.30}$$

$$I_3^n(2,3;0,X,Y) = -\frac{r_\Gamma}{\epsilon}\,\frac{1}{2(1-2\epsilon)}\,H_2(Y,X,-\epsilon) \tag{B.31}$$

$$\tag{B.32}$$

with three Feynman parameters:

$$I_3^n(1,1,1;0,0,X) = -\frac{r_\Gamma}{\epsilon}\,\frac{2-\epsilon}{2(3-2\epsilon)(1-2\epsilon)}\,H_0(X,-\epsilon) \tag{B.33}$$

$$I_3^n(2,2,2;0,0,X) = \frac{r_\Gamma}{\epsilon^2}\,\frac{3}{(1-\epsilon)(3-2\epsilon)(1-2\epsilon)}\,H_0(X,-\epsilon) \tag{B.34}$$

$$I_3^n(3,3,3;0,0,X) = -\frac{r_\Gamma}{\epsilon}\,\frac{2-\epsilon}{2(3-2\epsilon)(1-2\epsilon)}\,H_0(X,-\epsilon) \tag{B.35}$$

$$I_3^n(1,1,2;0,0,X) = -\frac{r_\Gamma}{\epsilon}\,\frac{1}{2(3-2\epsilon)(1-2\epsilon)}\,H_0(X,-\epsilon) \tag{B.36}$$

$$I_3^n(1,2,2;0,0,X) = -\frac{r_\Gamma}{\epsilon}\,\frac{1}{(1-\epsilon)(3-2\epsilon)(1-2\epsilon)}\,H_0(X,-\epsilon) \tag{B.37}$$

$$I_3^n(1,1,3;0,0,X) = r_\Gamma\,\frac{1}{2(3-2\epsilon)(1-2\epsilon)}\,H_0(X,-\epsilon) \tag{B.38}$$

$$I_3^n(2,2,3;0,0,X) = -\frac{r_\Gamma}{\epsilon}\,\frac{1}{(1-\epsilon)(3-2\epsilon)(1-2\epsilon)}\,H_0(X,-\epsilon) \tag{B.39}$$

$$I_3^n(1,3,3;0,0,X) = r_\Gamma\,\frac{1}{2(3-2\epsilon)(1-2\epsilon)}\,H_0(X,-\epsilon) \tag{B.40}$$

$$I_3^n(2,3,3;0,0,X) = -\frac{r_\Gamma}{\epsilon}\,\frac{1}{2(3-2\epsilon)(1-2\epsilon)}\,H_0(X,-\epsilon) \tag{B.41}$$

$$I_3^n(1,2,3;0,0,X) = r_\Gamma\,\frac{1}{2(3-2\epsilon)(1-2\epsilon)}\,H_0(X,-\epsilon) \tag{B.42}$$

$$I_3^n(1,1,1;0,X,Y) = \frac{r_\Gamma}{\epsilon^2}\,\frac{(3-\epsilon)(2-\epsilon)}{2(3-2\epsilon)(1-2\epsilon)}\,H_4(X,Y,-\epsilon) \tag{B.43}$$

$$I_3^n(2,2,2;0,X,Y) = \frac{r_\Gamma}{\epsilon^2}\,\frac{(3-\epsilon)(2-\epsilon)}{2(3-2\epsilon)(1-2\epsilon)}\,H_4(Y,X,-\epsilon) \tag{B.44}$$

$$I_3^n(3,3,3;0,X,Y) = -\frac{r_\Gamma}{\epsilon}\,\frac{(2-\epsilon)}{2(3-2\epsilon)(1-2\epsilon)}\,H_1(X,Y,-\epsilon) \tag{B.45}$$

$$I_3^n(1,1,2;0,X,Y) = \frac{r_\Gamma}{\epsilon^2}\,\frac{(3-\epsilon)(2-\epsilon)}{2(3-2\epsilon)(1-2\epsilon)}\,(H_3(X,Y,-\epsilon)-H_4(X,Y,-\epsilon)) \tag{B.46}$$

$$I_3^n(1,2,2;0,X,Y) = \frac{r_\Gamma}{\epsilon^2}\,\frac{(3-\epsilon)(2-\epsilon)}{2(3-2\epsilon)(1-2\epsilon)}\,(H_3(Y,X,-\epsilon)-H_4(Y,X,-\epsilon)) \tag{B.47}$$

$$I_3^n(1,1,3;0,X,Y) = -\frac{r_\Gamma}{\epsilon}\,\frac{(2-\epsilon)}{2(3-2\epsilon)(1-2\epsilon)}\,H_3(X,Y,-\epsilon) \tag{B.48}$$

$$I_3^n(2,2,3;0,X,Y) = -\frac{r_\Gamma}{\epsilon}\,\frac{(2-\epsilon)}{2(3-2\epsilon)(1-2\epsilon)}\,H_3(Y,X,-\epsilon) \tag{B.49}$$

$$I_3^n(1,3,3;0,X,Y) = -\frac{r_\Gamma}{\epsilon}\,\frac{(1-\epsilon)}{2(3-2\epsilon)(1-2\epsilon)}\,H_2(X,Y,-\epsilon) \tag{B.50}$$

$$I_3^n(2,3,3;0,X,Y) = -\frac{r_\Gamma}{\epsilon}\,\frac{(1-\epsilon)}{2(3-2\epsilon)(1-2\epsilon)}\,H_2(Y,X,-\epsilon) \tag{B.51}$$

$$I_3^n(1,2,3;0,X,Y) = -\frac{r_\Gamma}{\epsilon}\,\frac{(2-\epsilon)}{2(3-2\epsilon)(1-2\epsilon)}\,\left(H_2(X,Y,-\epsilon)-H_3(X,Y,-\epsilon)\right) \tag{B.52}$$

Higher dimensional 3-point integrals:

$$I_3^{n+2}(0,0,X) = \frac{r_\Gamma}{\epsilon}\frac{1}{2(1-\epsilon)(1-2\epsilon)}\,H_0(X,1-\epsilon) \tag{B.53}$$

$$I_3^{n+2}(0,X,Y) = \frac{r_\Gamma}{\epsilon}\frac{1}{2(1-\epsilon)(1-2\epsilon)}\,H_1(X,Y,1-\epsilon) \tag{B.54}$$

$$I_3^{n+2}(1;0,0,X) = \frac{r_\Gamma}{\epsilon}\frac{1}{2(3-2\epsilon)(1-2\epsilon)}\,H_0(X,1-\epsilon) \tag{B.55}$$

$$I_3^{n+2}(2;0,0,X) = \frac{r_\Gamma}{\epsilon}\frac{1}{2(1-\epsilon)(3-2\epsilon)(1-2\epsilon)}\,H_0(X,1-\epsilon) \tag{B.56}$$

$$I_3^{n+2}(3;0,0,X) = \frac{r_\Gamma}{\epsilon}\frac{1}{2(3-2\epsilon)(1-2\epsilon)}\,H_0(X,1-\epsilon) \tag{B.57}$$

$$I_3^{n+2}(1;0,X,Y) = \frac{r_\Gamma}{\epsilon}\frac{(2-\epsilon)}{2(1-\epsilon)(3-2\epsilon)(1-2\epsilon)}\,H_2(X,Y,1-\epsilon) \tag{B.58}$$

$$I_3^{n+2}(2;0,X,Y) = \frac{r_\Gamma}{\epsilon}\frac{(2-\epsilon)}{2(1-\epsilon)(3-2\epsilon)(1-2\epsilon)}\,H_2(Y,X,1-\epsilon) \tag{B.59}$$

$$I_3^{n+2}(3;0,X,Y) = \frac{r_\Gamma}{\epsilon}\frac{1}{2(3-2\epsilon)(1-2\epsilon)}\,H_1(X,Y,1-\epsilon) \tag{B.60}$$

The functions H_0, H_1, H_2, H_3 and H_4 are given by:

$$H_0(X,\alpha) = \frac{\bar{X}^\alpha}{X} \tag{B.61}$$

$$H_1(X,Y,\alpha) = \frac{\bar{X}^\alpha-\bar{Y}^\alpha}{X-Y} \tag{B.62}$$

$$H_2(X,Y,\alpha) = \frac{\bar{Y}^\alpha}{Y-X}+\frac{1}{1+\alpha}\,\frac{\bar{Y}^{1+\alpha}-\bar{X}^{1+\alpha}}{(Y-X)^2} \tag{B.63}$$

$$H_3(X,Y,\alpha) = \frac{\bar{Y}^\alpha}{Y-X}+\frac{2}{1+\alpha}\,\frac{\bar{Y}^{1+\alpha}}{(Y-X)^2}+\frac{2}{(1+\alpha)\,(2+\alpha)}\,\frac{\bar{Y}^{2+\alpha}-\bar{X}^{2+\alpha}}{(Y-X)^3} \tag{B.64}$$

$$\begin{aligned} H_4(X,Y,\alpha) &= \frac{\bar{Y}^\alpha}{Y-X}+\frac{3}{1+\alpha}\,\frac{\bar{Y}^{1+\alpha}}{(Y-X)^2}+\frac{6}{(1+\alpha)\,(2+\alpha)}\,\frac{\bar{Y}^{2+\alpha}}{(Y-X)^3} \\ &\quad +\frac{6}{(1+\alpha)\,(2+\alpha)\,(3+\alpha)}\,\frac{\bar{Y}^{3+\alpha}-\bar{X}^{3+\alpha}}{(Y-X)^4} \end{aligned} \tag{B.65}$$

$$\bar{X} = -X - i\,\delta$$

All these functions have a regular behaviour when $X=Y$. It is for this reason that we want to keep them as they are. If the coefficient in front of them is proportional to $(X-Y)$, then

they can be reduced using the following properties:

$$
\begin{aligned}
(Y-X)\,H_1(X,Y,\alpha) &= \bar{Y}^{\alpha}-\bar{X}^{\alpha} && \text{(B.66)}\\
(Y-X)\,H_2(X,Y,\alpha) &= \frac{\alpha}{1+\alpha}\,\bar{Y}^{\alpha}-\frac{1}{1+\alpha}\,X\,H_1(X,Y,\alpha) && \text{(B.67)}\\
(Y-X)\,H_3(X,Y,\alpha) &= \frac{\alpha}{2+\alpha}\,\bar{Y}^{\alpha}-\frac{2}{2+\alpha}\,X\,H_2(X,Y,\alpha) && \text{(B.68)}\\
(Y-X)\,H_4(X,Y,\alpha) &= \frac{\alpha}{3+\alpha}\,\bar{Y}^{\alpha}-\frac{3}{3+\alpha}\,X\,H_3(X,Y,\alpha) && \text{(B.69)}
\end{aligned}
$$

$$
\begin{aligned}
H_1(Y,X,\alpha) &= H_1(X,Y,\alpha)\\
H_2(Y,X,\alpha) &= H_1(X,Y,\alpha)-H_2(X,Y,\alpha)\\
H_3(Y,X,\alpha) &= H_3(X,Y,\alpha)-2\,H_2(X,Y,\alpha)+H_1(X,Y,\alpha)\\
H_4(Y,X,\alpha) &= -H_4(X,Y,\alpha)+3\,H_3(X,Y,\alpha)-3\,H_2(X,Y,\alpha)+H_1(X,Y,\alpha) \qquad \text{(B.70)}
\end{aligned}
$$

We use $n = 4 - 2\epsilon$ where $\epsilon < 0$ in the infrared region.

C Proof of the absence of higher dimensional integrals and Gram determinants for $N = 5$

In the next subsection, we will show that the coefficient multiplying the higher dimensional five-point integrals is of order ϵ. We will repeatedly use the fact that for general 5-point kinematics, the metric tensor in 4 dimensions is expressible by a tensor product of external vectors. Many simplifications will occur by neglecting terms of "$\mathcal{O}(\epsilon)$". For scalar quantities it is clear what that means. For *tensors*, we say that a tensorial structure is of $\mathcal{O}(\epsilon)$ if differences of tensors defined in n and in 4 dimensions occur. Contracting such differences with kinematical objects like external momenta, polarisation vectors or fermion currents will always lead finally to scalar quantities of $\mathcal{O}(\epsilon)$, which can be neglected in phenomenological applications at one loop.

The price to pay for the disappearance of the higher dimensional integrals is that inverse Gram determinants ($1/B$) are reintroduced explicitly. In subsection C.2, we show how these spurious divergences cancel out analytically. This will lead us to a representation of 5-point functions which is free from higher dimensional 5-point integrals *and* $1/B$ terms. The corresponding form factors are listed in section 6 of the main text.

C.1 The fate of higher dimensional integrals

For $N = 5$, the following relation, shown in appendix D.3,

$$
\mathcal{T}_{[4]}{}^{\mu\,\nu}_{a\,b} = 2\,\frac{\mathcal{V}^{\mu}_{a}\,\mathcal{V}^{\nu}_{b}}{B} \qquad \text{(C.71)}
$$

will enable us to remove the higher dimensional five-point integrals. The tensors $\mathcal{V}^{\mu}_{a}$ and $\mathcal{T}^{\mu\,\nu}_{a\,b}$ are defined in eqs. (51) and (53).

For rank zero and rank one, it is trivial to see that the coefficient of I_5^{n+2} is of order ϵ. For rank 0, one has, according to eqs. (24) and (29):

$$I_5^n(S) = \sum_{j\in S} b_j \, I_4^n(S\setminus\{j\}) - (4-n) B \, I_5^{n+2}(S) \; . \tag{C.72}$$

For rank 1, one obtains by application of (46) :

$$I_5^{n,\,\mu}(a,S) = -\sum_{j\in S} \mathcal{C}^{\mu}_{j\,a} \, I_4^n(S\setminus\{j\}) + (4-n)\, \mathcal{V}^{\mu}_{a} \, I_5^{n+2}(S) \; . \tag{C.73}$$

For higher rank ($r \geq 2$), we prove by induction on r that the coefficient of the higher dimensional five-point integrals is of order ϵ.

In step one we show that the assumption is true for $r = 2$. Indeed, in the case of rank 2, direct application of (46) leads to

$$\begin{aligned} I_5^{n,\,\mu\nu}(a_1,a_2;S) &= \int d\bar{k} \, \frac{q^{\mu}_{a_1}\, q^{\nu}_{a_2}}{\prod_{i\in S}(q_i^2 - m_i^2 + i\delta)} \\ &= -\frac{1}{2}\, \mathcal{T}^{\mu\nu}_{a_1 a_2} \, I_5^{n+2}(S) + (3-n) \sum_{i\in S} I_5^{n+2}(i\,;S)\, \Delta^{\mu}_{a_1\,i}\, \mathcal{V}^{\nu}_{a_2} \\ &\quad -\sum_{j\in S} \mathcal{C}^{\nu}_{j\,a_2}\, I_4^{n,\,\mu}(a_1\,;S\setminus\{j\}) \; . \end{aligned} \tag{C.74}$$

The integral $I_5^{n+2}(i\,;S)$ with Feynman parameter z_i in the numerator can be expressed in terms of $I_5^{n+2}(S)$. For this purpose, we contract both sides of eq. (C.74) with $\Delta^{\nu}_{b\,c}$, where b and c are arbitrary labels in S. Using

$$q^{\mu}_{a}\, \Delta_{b\,c\,\mu} = \frac{1}{2} \left(q_b^2 - m_b^2 - [\, q_c^2 - m_c^2 \,] - \mathcal{S}_{a\,b} + \mathcal{S}_{a\,c} \right) \tag{C.75}$$

$$\mathcal{T}^{\mu\,\nu}_{a\,b}\, \Delta_{c\,d\,\mu} = (\mathcal{S}_{ac} - \mathcal{S}_{ad})\, \mathcal{V}^{\nu}_{b} \tag{C.76}$$

$$\mathcal{V}^{\mu}_{a}\, \Delta_{c\,d\,\mu} = \frac{B}{2}\, (\mathcal{S}_{ac} - \mathcal{S}_{ad}) \tag{C.77}$$

$$\mathcal{C}^{\mu}_{j\,a}\, \Delta_{c\,d\,\mu} = \frac{1}{2}\, (\delta_{jd} - \delta_{jc} + b_j\, [\, \mathcal{S}_{ac} - \mathcal{S}_{ad}\,]) \tag{C.78}$$

leads to

$$\begin{aligned} \sum_{i\in S} I_5^{n+2}(i\,;S)\, \Delta^{\mu}_{a\,i} &= \frac{1}{B} \left\{ -\mathcal{V}^{\mu}_{a}\, I_5^{n+2}(S) \right. \\ &\quad \left. + \frac{1}{3-n} \sum_{i\in S} \left(b_i\, I_4^{n,\,\mu}(a, S\setminus\{i\}) + \mathcal{C}^{\mu}_{i\,a}\, I_4^n(S\setminus\{i\}) \right) \right\} . \end{aligned} \tag{C.79}$$

The insertion of the latter into eq. (C.74) yields

$$I_5^{n,\,\mu\nu}(a_1,a_2;S) = \left\{ -\frac{1}{2}\, \mathcal{T}^{\mu\nu}_{a_1 a_2} + (n-3)\, \frac{\mathcal{V}^{\mu}_{a_1}\, \mathcal{V}^{\nu}_{a_2}}{B} \right\} I_5^{n+2}(S)$$

$$
\begin{aligned}
&+\sum_{i\in S}\left(\frac{\mathcal{V}^{\nu}_{a_2}}{B}\,b_i-\mathcal{C}^{\nu}_{i\,a_2}\right)\,I_4^{n,\,\mu}(a_1\,;S\setminus\{i\})\\
&+\frac{\mathcal{V}^{\nu}_{a_2}}{B}\,\sum_{i\in S}\mathcal{C}^{\mu}_{i\,a_1}\,I_4^{n}(S\setminus\{i\})\;. \qquad \text{(C.80)}
\end{aligned}
$$

Using now eq. (C.73) for the last term in eq. (C.80) results in

$$
\begin{aligned}
I_5^{n,\,\mu\nu}(a_1,a_2;S) \;=\; & -\frac{1}{2}\left\{\mathcal{T}^{\mu\nu}_{a_1a_2}-\frac{2\,\mathcal{V}^{\mu}_{a_1}\,\mathcal{V}^{\nu}_{a_2}}{B}\right\}\,I_5^{n+2}(S)\\
&+\sum_{i\in S}\left(\frac{\mathcal{V}^{\nu}_{a_2}}{B}\,b_i-\mathcal{C}^{\nu}_{i\,a_2}\right)\,I_4^{n,\,\mu}(a_1\,;S\setminus\{i\})\\
&-\frac{\mathcal{V}^{\nu}_{a_2}}{B}\,I_5^{n,\,\mu}(a_1\,;S)\;. \qquad \text{(C.81)}
\end{aligned}
$$

Now we can use eq. (C.71) to see that the coefficient of I_5^{n+2} in eq. (C.81) is indeed of order ϵ.

Let us now assume that for rank $r-1$, all the higher dimensional integrals involve an $\mathcal{O}(\epsilon)$ tensor of the type $\mathcal{T}^{\mu\nu}_{a_1a_2}-2\,\mathcal{V}^{\mu}_{a_1}\,\mathcal{V}^{\nu}_{a_2}/B$, and show that this is also the case for rank r. Firstly, we carry out the momentum integration in the first term of eq. (46), using the shift (25) and eqs. (47), (52). This leads to[7]:

$$
\begin{aligned}
I_N^{n,\,\mu_1\ldots\mu_r}(a_1,\ldots,a_r;S) = -\sum_{j\in S}\mathcal{C}^{\mu_r}_{j\,a_r}\int d\bar{k}\,\frac{(q_j^2-m_j^2)\;q_{a_1}^{\mu_1}\ldots q_{a_{r-1}}^{\mu_{r-1}}}{\prod_{i\in S}(q_i^2-m_i^2+i\delta)} \qquad \text{(C.82)}\\
+\int d\vec{Z}\int d\bar{l}\,\frac{\left[l_\nu\,\left(\mathcal{T}^{\mu_r\nu}_{a_rd}+2\,\mathcal{V}^{\mu_r}_{a_r}\sum_{i\in S}z_i\Delta^{\nu}_{d\,i}\right)+\mathcal{V}^{\mu_r}_{a_r}\,(l^2+R^2)\right]\;\tilde{q}_{a_1}^{\mu_1}\ldots\tilde{q}_{a_{r-1}}^{\mu_{r-1}}}{(l^2-R^2)^N}\;,
\end{aligned}
$$

where

$$
d\vec{Z} \;=\; \Gamma(N)\,\prod_{i\in S}dz_i\,\delta(1-\sum_{l=1}^{N}z_l)
$$

and the $\tilde{q}_a$ denote the q_a-vectors in terms of the shifted loop momentum l, given by eq. (47).

Secondly, we contract eq. (C.82) with $\Delta_{b\,c\,\mu_r}$, where $b,c\in S$ are arbitrary, using (C.75) for the left-hand side and (C.76) to (C.78) for the right-hand side. This yields

$$
\begin{aligned}
\int d\bar{k}\,\frac{q_{a_1}^{\mu_1}\ldots q_{a_{r-1}}^{\mu_{r-1}}}{\prod_{i\in S}(q_i^2-m_i^2+i\delta)} = \sum_{j\in S}b_j\,\int d\bar{k}\,\frac{(q_j^2-m_j^2)\;q_{a_1}^{\mu_1}\ldots q_{a_{r-1}}^{\mu_{r-1}}}{\prod_{i\in S}(q_i^2-m_i^2+i\delta)} \qquad \text{(C.83)}\\
-\int d\vec{Z}\,\int d\bar{l}\,\frac{\left[2\,l_\nu\left(\mathcal{V}^{\nu}_{d}+B\,\sum_{i\in S}z_i\,\Delta^{\nu}_{d\,i}\right)+B\,(l^2+R^2)\right]\;\tilde{q}_{a_1}^{\mu_1}\ldots\tilde{q}_{a_{r-1}}^{\mu_{r-1}}}{(l^2-R^2)^N}\;.
\end{aligned}
$$

Using now eq. (C.83) to replace the term containing $\sum_{i\in S}z_i\,\Delta^{\nu}_{d\,i}$ in (C.82), we obtain:

$$
I_N^{n,\,\mu_1\ldots\mu_r}(a_1,\ldots,a_r\,;S) \;=\; \sum_{j\in S}\left(\frac{\mathcal{V}^{\mu_r}_{a_r}\,b_j}{B}-\mathcal{C}^{\mu_r}_{j\,a_r}\right)\,\int d\bar{k}\,\frac{(q_j^2-m_j^2)\;q_{a_1}^{\mu_1}\ldots q_{a_{r-1}}^{\mu_{r-1}}}{\prod_{i\in S}(q_i^2-m_i^2+i\delta)}
$$

[7]Here we keep N arbitrary to show that the relations derived are not only valid for $N=5$.

$$- \frac{\mathcal{V}^{\mu_r}_{a_r}}{B} I_N^{n,\mu_1\ldots\mu_{r-1}}(a_1,\ldots,a_{r-1}\,;S)$$
$$+ \int d\vec{Z} \int d\bar{l}\, \frac{l_\nu\, \left(\mathcal{T}^{\mu_r\nu}_{a_r d} - 2\,\mathcal{V}^{\mu_r}_{a_r}\mathcal{V}^{\nu}_{d}/B\right)\, \tilde{q}^{\mu_1}_{a_1}\ldots\tilde{q}^{\mu_{r-1}}_{a_{r-1}}}{(l^2-R^2)^N}\,. \quad \text{(C.84)}$$

Specifying $N=5$, we can see from eq. (C.84) that a rank r 5-point integral can be written as the sum of 4-point integrals of rank $r-1$ plus a 5-point integral of rank $r-1$ plus a term which potentially generates higher dimensional 5-point integrals but is proportional to $\mathcal{T}^{\mu_r\mu_k}_{a_r a_k} - 2\,\mathcal{V}^{\mu_r}_{a_r}\,\mathcal{V}^{\mu_k}_{a_k}/B$, which is of order ϵ.

A few comments are in order here.

(i) Eq. (C.83) defines another way of reducing the tensor integrals, by which a rank r N-point integral is expressed as an infrared finite part plus a sum of rank r $(N-1)$-point integrals. Contrarily to eq. (C.82), eq. (C.83) reduces the number of propagators but not the rank. For this reason the way used in section 4 is preferable.

(ii) In the case where $\mathcal{S}$ is not invertible, the derivation is equally valid, as eq. (C.78) still holds in this case. This can be easily seen by contracting eq. (C.82) for $r=1$ with $\Delta_{bc\,\mu}$. One obtains

$$\sum_{j\in S} \mathcal{C}^{\mu}_{ja}\Delta_{bc\,\mu}\, I^n_{N-1}(S\setminus\{j\}) = \frac{1}{2}\left(I^n_{N-1}(S\setminus\{c\}) - I^n_{N-1}(S\setminus\{b\}) + (\mathcal{S}_{ab}-\mathcal{S}_{ac})\, I^n_N(S)\right)\,.$$

Now we can use $I^n_N(S) = \sum_{j\in S} b_j\, I^n_{N-1}(S\backslash\{j\})$, which is always valid if $B=\sum_{j\in S} b_j = 0$, and fulfilled to order ϵ for $N=5$. The reduction coefficients b_j can be constructed as in section 4.1, if $\mathcal{S}$ is not invertible. Thus one obtains

$$\sum_{j\in S} \mathcal{C}^{\mu}_{ja}\Delta_{bc\,\mu}\, I^n_{N-1}(S\setminus\{j\}) = \frac{1}{2}\sum_{j\in S}\, I^n_{N-1}(S\setminus\{j\})\Big[\delta_{jc} - \delta_{jb} + b_j\,(\mathcal{S}_{ab}-\mathcal{S}_{ac})\Big]\,. \quad \text{(C.85)}$$

which leads to eq. (C.78) without using the inverse of $\mathcal{S}$.

(iii) A similar proof is given in eqs. (34) to (37) of [50], but now we have a shift invariant formulation, and all coefficients are expressed in terms of quantities containing only $\mathcal{S}^{-1}_{ij}$ instead of H_{ij} (the pseudo-inverse of the Gram matrix G).

In summary, we have shown that

$$\begin{aligned}
&I_5^{n,\,\mu_1\mu_2\ldots\mu_r}(a_1,a_2,\cdots,a_r\,;S) = \\
&\quad \frac{\mathcal{V}^{\mu_r}_{a_r}}{B}\left[-I_5^{n,\,\mu_1\cdots\mu_{r-1}}(a_1,\ldots,a_{r-1}\,;S) + \sum_{j\in S} b_j\, I_4^{n,\,\mu_1\ldots\mu_{r-1}}(a_1,\ldots,a_{r-1}\,;S\setminus\{j\})\right] \\
&\quad -\sum_{j\in S} \mathcal{C}^{\mu_r}_{j\,a_r}\, I_4^{n,\,\mu_1\cdots\mu_{r-1}}(a_1,\ldots,a_{r-1}\,;S\setminus\{j\}) \qquad \text{(C.86)}
\end{aligned}$$

In eq. (C.86), inverse Gram determinants ($1/B$) have been explicitly reintroduced in the term

$$F^{\mu_1\cdots\mu_r}(a_1,\ldots,a_r) = \frac{\mathcal{V}^{\mu_r}_{a_r}}{B}\left[-I_5^{n,\mu_1\cdots\mu_{r-1}}(a_1,\ldots,a_{r-1}\,;S) + \sum_{j\in S} b_j\, I_4^{n,\mu_1\cdots\mu_{r-1}}(a_1,\ldots,a_{r-1}\,;S\setminus\{j\})\right] \quad \text{(C.87)}$$

We show in the next section how they drop out.

C.2 Cancellation of 1/B terms for $N = 5$

In this section we will prove by induction that $F^{\mu_1\cdots\mu_r}(a_1,\ldots,a_r)$, defined in eq. (C.87), is in fact free from $1/B$ terms if the factor $\mathcal{V}^{\mu_r}_{a_r}/B$ is combined with the expressions inside the square bracket.

To this end, we first use eq. (C.86) for rank $r-1$

$$\begin{aligned} I_5^{n,\mu_1\mu_2\cdots\mu_{r-1}}&(a_1,a_2,\cdots,a_{r-1}\,;S) = \\ &F^{\mu_1\cdots\mu_{r-1}}(a_1,\ldots,a_{r-1}) - \sum_{j\in S}\mathcal{C}^{\mu_{r-1}}_{j\,a_{r-1}}\, I_4^{n,\mu_1\cdots\mu_{r-2}}(a_1,\ldots,a_{r-2}\,;S\setminus\{j\}) \end{aligned} \quad \text{(C.88)}$$

and insert the above equation into eq. (C.87), leading to

$$\begin{aligned} I_5^{n,\mu_1\mu_2\cdots\mu_r}&(a_1,a_2,\cdots,a_r\,;S) = \\ &\frac{\mathcal{V}^{\mu_r}_{a_r}}{B}\Bigg[-F^{\mu_1\cdots\mu_{r-1}}(a_1,\cdots,a_{r-1}) \\ &+\sum_{j\in S}\mathcal{C}^{\mu_{r-1}}_{j\,a_{r-1}}\, I_4^{n,\mu_1\cdots\mu_{r-2}}(a_1,\ldots,a_{r-2}\,;S\setminus\{j\}) + \sum_{j\in S} b_j\, I_4^{n,\mu_1\cdots\mu_{r-1}}(a_1,\ldots,a_{r-1}\,;S\setminus\{j\})\Bigg] \\ &-\sum_{j\in S}\mathcal{C}^{\mu_r}_{j\,a_r}\, I_4^{n,\mu_1\cdots\mu_{r-1}}(a_1,\ldots,a_{r-1}\,;S\setminus\{j\}) \end{aligned} \quad \text{(C.89)}$$

Actually, the iteration of eq. (C.86) performed in eq. (C.89) singles out the pair of indices (a_{r-1},μ_{r-1}), which hides the manifest symmetry of $I_5^{n,\mu_1\cdots\mu_{r-1}}(a_1,\cdots,a_{r-1}\,;S)$ with respect to all pairs (a_1,μ_1), $\cdots$, (a_{r-1},μ_{r-1}). However, the explicit cancellation of $1/B$ terms relies on this symmetry. Therefore we introduce a symmetrisation operator Ξ_s which acts as follows: If $W^{\mu_1\ldots\mu_s}(a_1,\ldots,a_s)$ is a tensor which already is symmetric with respect to the $s-1$ first indices, Ξ_s is defined by

$$\Xi_s[W^{\mu_1\ldots\mu_s}(a_1,\ldots,a_s)] \;=\; \frac{1}{s}\left[W^{\mu_1\ldots\mu_s}(a_1,\ldots,a_s) + \text{c. p. }\right], \quad \text{(C.90)}$$

where "c. p." means the sum over cyclic permutations of (a_1,μ_1), $\cdots$, (a_s,μ_s). Thus we can write eq. (C.89) as

$$\begin{aligned} I_5^{n,\mu_1\mu_2\ldots\mu_r}(a_1,a_2,\cdots,a_r\,;S) \;=\; &\frac{\mathcal{V}^{\mu_r}_{a_r}}{B}\left[-\Xi_{r-1}F^{\mu_1\cdots\mu_{r-1}}(a_1,\cdots,a_{r-1}) + Q^{\mu_1\ldots\mu_{r-1}}\right] \\ &-\sum_{j\in S}\mathcal{C}^{\mu_r}_{j\,a_r}\, I_4^{n,\mu_1\cdots\mu_{r-1}}(a_1,\ldots,a_{r-1}\,;S\setminus\{j\})\,, \end{aligned} \quad \text{(C.91)}$$

where

$$
\begin{aligned}
Q^{\mu_1\cdots\mu_{r-1}} &= \sum_{j\in S}\Bigg\{ \Xi_{r-1}\left[\mathcal{C}^{\mu_{r-1}}_{j\,a_{r-1}}\, I_4^{n,\mu_1\cdots\mu_{r-2}}(a_1,\ldots,a_{r-2}\,;S\setminus\{j\})\right] \\
&\qquad + b_j\, I_4^{n,\mu_1\cdots\mu_{r-1}}(a_1,\ldots,a_{r-1}\,;S\setminus\{j\})\Bigg\} \qquad \text{(C.92)}
\end{aligned}
$$

In order to show that the $1/B$ terms in the first line of eq. (C.91) drop out, let us first consider $Q^{\mu_1\cdots\mu_{r-1}}$. Using eq. (C.83) with $N=4$ for the first term and eq. (C.82) with $N=4$ for the second term in eq. (C.92), we obtain:

$$
\begin{aligned}
Q^{\mu_1\cdots\mu_{r-1}} &= Q_1^{\mu_1\cdots\mu_{r-1}} + \sum_{j\in S}\Bigg(\Xi_{r-1}[(b_j\,\mathcal{V}^{\mu_{r-1}}_{a_{r-1}} - B\,\mathcal{C}^{\mu_{r-1}}_{j\,a_{r-1}})\,Q_{2\,j}^{\mu_1\cdots\mu_{r-2}}] \qquad \text{(C.93)} \\
&+ \sum_{k\in S\setminus\{j\}} \Xi_{r-1}\int d\bar{k}\,\frac{(q_j^2-m_j^2)\,(q_k^2-m_k^2)\,q_{a_1}^{\mu_1}\ldots q_{a_{r-2}}^{\mu_{r-2}}}{\prod_{i\in S}(q_i^2-m_i^2+i\delta)}\,\left[\mathcal{C}^{\mu_{r-1}}_{j\,a_{r-1}}\,b_k - b_j\,\mathcal{C}^{\mu_{r-1}}_{k\,a_{r-1}}\right]\Bigg)
\end{aligned}
$$

where

$$
\begin{aligned}
Q_1^{\mu_1\cdots\mu_{r-1}} &= \sum_{j\in S}\Xi_{r-1}\left[\left(b_j\,\mathcal{T}^{\mu_{r-1}\,\nu}_{a_{r-1}\,d} - 2\,\mathcal{C}^{\mu_{r-1}}_{j\,a_{r-1}}\,\mathcal{V}^{\nu}_{d}\right)\int d\vec{Z}\int d\bar{l}\,\frac{l_\nu\,\tilde{q}_{a_1}^{\mu_1}\cdots\tilde{q}_{a_{r-2}}^{\mu_{r-2}}}{(l^2-R_j^2)^4} \qquad \text{(C.94)}\right. \\
Q_{2\,j}^{\mu_1\cdots\mu_{r-2}} &= \int d\vec{Z}\int d\bar{l}\,\frac{\tilde{q}_{a_1}^{\mu_1}\cdots\tilde{q}_{a_{r-2}}^{\mu_{r-2}}}{(l^2-R_j^2)^4}\left(2\sum_{i=1}^{4} z_i\,l\cdot\Delta_{d\,i} + l^2 + R_j^2\right) \qquad \text{(C.95)} \\
R_j^2 &= -\frac{1}{2}\sum_{k,l\in S\setminus\{j\}} z_k\,\mathcal{S}^{\{j\}}{}_{k\,l}\,z_l\ ,
\end{aligned}
$$

and the identities (D.145), (D.146) and (D.147) have been used. The last term of eq. (C.93) vanishes due to antisymmetry with respect to j and k. Therefore, the first line in eq. (C.91) can be rewritten as

$$
\frac{\mathcal{V}^{\mu_r}_{a_r}}{B}\left[-\Xi_{r-1}\,F^{\mu_1\cdots\mu_{r-1}} + Q_1^{\mu_1\cdots\mu_{r-1}} + \sum_{j\in S}\Xi_{r-1}[(b_j\,\mathcal{V}^{\mu_{r-1}}_{a_{r-1}} - B\,\mathcal{C}^{\mu_{r-1}}_{j\,a_{r-1}})\,Q_{2\,j}^{\mu_1\cdots\mu_{r-2}}]\right] \qquad \text{(C.96)}
$$

The term multiplying $Q_{2\,j}^{\mu_1\cdots\mu_{r-2}}$ involves $(b_j\,\mathcal{V}^{\mu_i}_{a_i} - B\,\mathcal{C}^{\mu_i}_{j\,a_i})\,\mathcal{V}^{\mu_r}_{a_r}/B$, and due to eq. (C.71) we have

$$
\frac{\mathcal{V}^{\mu_r}_{a_r}}{B}\,(b_j\,\mathcal{V}^{\mu_i}_{a_i} - B\,\mathcal{C}^{\mu_i}_{j\,a_i}) \;=\; \frac{1}{2}\,b_j\,\mathcal{T}_{[4]}{}^{\mu_i\,\mu_r}_{a_i\,a_r} - \mathcal{V}^{\mu_r}_{a_r}\,\mathcal{C}^{\mu_i}_{j\,a_i}\ , \qquad \text{(C.97)}
$$

explicitly free from $1/B$ terms.

For the remaining contribution, we will show that $Q_1^{\mu_1\cdots\mu_{r-1}} = \Xi_{r-1}\,F^{\mu_1\cdots\mu_{r-1}} + \mathcal{O}(\epsilon)$. More in detail, we will prove by induction that

$$
F^{\mu_1\cdots\mu_{r-1}} = \sum_{j\in S}\frac{\mathcal{V}^{\mu_{r-1}}_{a_{r-1}}}{B}\,\Xi_{r-2}\left[\left(b_j\,\mathcal{V}^{\mu_{r-2}}_{a_{r-2}} - B\,\mathcal{C}^{\mu_{r-2}}_{j\,a_{r-2}}\right)\,Q_{2\,j}^{\mu_1\cdots\mu_{r-3}}\right] + \mathcal{O}(\epsilon) \qquad \text{(C.98)}
$$

and show by direct calculation that we also have

$$Q_1^{\mu_1\cdots\mu_{r-1}} = \Xi_{r-1}\left[\sum_{j\in S}\frac{\mathcal{V}^{\mu_{r-1}}_{a_{r-1}}}{B}\,\Xi_{r-2}\left[\left(b_j\,\mathcal{V}^{\mu_{r-2}}_{a_{r-2}} - B\,\mathcal{C}^{\mu_{r-2}}_{j\,a_{r-2}}\right)\,Q_{2\,j}^{\mu_1\cdots\mu_{r-3}}\right]\right] + \mathcal{O}(\epsilon) \tag{C.99}$$

Eq. (C.99) is established in subsection C.3 for $r = 1,\ldots,5$. The case $r > 5$ will never be needed, but the mechanism is exactly the same.

Let us now show eq. (C.98) by induction. The induction start is $r = 2$, because for $r = 1$ the absence of $1/B$ terms is trivial. For $r = 2$, we combine eqs. (C.73) and (C.88) to obtain

$$F^{\mu_1}(a_1) = (4-n)\mathcal{V}^{\mu_1}_{a_1}\,I_5^{n+2}(S) = \mathcal{O}(\epsilon) \tag{C.100}$$

From eq. (C.94), we obtain

$$Q_1^{\mu_1} \;=\; \sum_{j\in S}\left(b_j\,\mathcal{T}^{\mu_1\,\nu}_{a_1\,d} - 2\,\mathcal{C}^{\mu_1}_{j\,a_1}\,\mathcal{V}^{\nu}_{d}\right)\int d\vec{Z}\int d\bar{l}\,\frac{l_\nu}{(l^2-R_j^2)^4} = 0 \tag{C.101}$$

Now let us assume that (C.98) is fulfilled for rank $r-1$. To prove the step $r-1 \to r$, we use eq. (C.91) for $I_5^{n,\mu_1\cdots\mu_r}(a_1,\cdots,a_r\,;S)$ and replace $F^{\mu_1\cdots\mu_{r-1}}$ by (C.98), which is true by the induction assumption, to obtain

$$\begin{aligned}
I_5^{n,\mu_1\cdots\mu_r}(a_1,\cdots,a_r\,;S) \;=\; & \frac{\mathcal{V}^{\mu_r}_{a_r}}{B}\Bigg(-\,\Xi_{r-1}\left\{\sum_{j\in S}\frac{\mathcal{V}^{\mu_{r-1}}_{a_{r-1}}}{B}\,\Xi_{r-2}\left[\left(b_j\,\mathcal{V}^{\mu_{r-2}}_{a_{r-2}} - B\,\mathcal{C}^{\mu_{r-2}}_{j\,a_{r-2}}\right)\,Q_{2\,j}^{\mu_1\cdots\mu_{r-3}}\right]\right\} \\
& + Q_1^{\mu_1\cdots\mu_{r-1}} + \sum_{j\in S}\Xi_{r-1}\left[\left(b_j\,\mathcal{V}^{\mu_{r-1}}_{a_{r-1}} - B\,\mathcal{C}^{\mu_{r-1}}_{j\,a_{r-1}}\right)\,Q_{2j}^{\mu_1\cdots\mu_{r-2}}\right]\Bigg) \\
& - \sum_{j\in S}\mathcal{C}^{\mu_r}_{j\,a_r}\,I_4^{n,\mu_1\cdots\mu_{r-1}}(a_1,\cdots,a_{r-1}\,;S\setminus\{j\})\,.
\end{aligned} \tag{C.102}$$

Comparing with eq. (C.99), we see that the term $\Xi_{r-1}\{\ldots\}$ in the first line of eq. (C.102) is equal to $Q_1^{\mu_1\cdots\mu_{r-1}}$. Comparing the remaining terms to eqs. (C.88) and (C.98) proves our assumption. Therefore, using eq. (C.97), we see that rank r 5-point integrals can be written as

$$\begin{aligned}
I_5^{n,\mu_1\cdots\mu_r}(a_1,\cdots,a_r\,;S) \;=\; & \sum_{j\in S}\Bigg\{\Xi_{r-1}\left[\left(\frac{1}{2}\,b_j\,\mathcal{T}_{[4]}{}^{\mu_{r-1}\,\mu_r}_{a_{r-1}\,a_r} - \mathcal{V}^{\mu_r}_{a_r}\,\mathcal{C}^{\mu_{r-1}}_{j\,a_{r-1}}\right)\,Q_{2\,j}^{\mu_1\cdots\mu_{r-2}}\right] \\
& - \mathcal{C}^{\mu_r}_{j\,a_r}\,I_4^{n,\mu_1\cdots\mu_{r-1}}(a_1,\cdots,a_{r-1}\,;S\setminus\{j\})\Bigg\} + \mathcal{O}(\epsilon)\,,
\end{aligned} \tag{C.103}$$

which is a combination of 4-point integrals.

C.3 Auxiliary relations

We do not have to compute 5-point integrals with rank bigger than five. Therefore it is sufficient to establish eq. (C.99) by direct calculation of $Q^{\mu_1\cdots\mu_{r-1}}$ and $Q_{2j}^{\mu_1\cdots\mu_{r-2}}$ for $r = 1,\ldots,5$. Using

the definition (C.94) and eqs. (D.145), (D.146) and (D.147), one gets:

$$Q_1^{\mu_1\,\mu_2} = -\frac{1}{2}\sum_{j\in S} D^{\mu_1\,\mu_2}_{a_1\,a_2}\, I_4^{n+2}(S\setminus\{j\}) \tag{C.104}$$

$$Q_1^{\mu_1\,\mu_2\,\mu_3} = -\frac{1}{3}\sum_{i,j\in S} I_4^{n+2}(i;S\setminus\{j\})\,(D^{\mu_2\,\mu_3}_{a_2\,a_3}\,\Delta^{\mu_1}_{a_1\,i} + D^{\mu_1\,\mu_3}_{a_1\,a_3}\,\Delta^{\mu_2}_{a_2\,i} + D^{\mu_1\,\mu_2}_{a_1\,a_2}\,\Delta^{\mu_3}_{a_3\,i}) \tag{C.105}$$

$$\begin{aligned} Q_1^{\mu_1\,\mu_2\,\mu_3\,\mu_4} = \; & \frac{1}{4}\sum_{j\in S}\Bigg[\frac{1}{2}\, I_4^{n+4}(S\setminus\{j\})\,(g^{\mu_1\,\mu_2}\,D^{\mu_3\,\mu_4}_{a_3\,a_4} + g^{\mu_1\,\mu_3}\,D^{\mu_2\,\mu_4}_{a_2\,a_4} + g^{\mu_1\,\mu_4}\,D^{\mu_2\,\mu_3}_{a_2\,a_3} \\ & + g^{\mu_2\,\mu_3}\,D^{\mu_1\,\mu_4}_{a_1\,a_4} + g^{\mu_2\,\mu_4}\,D^{\mu_1\,\mu_3}_{a_1\,a_3} + g^{\mu_3\,\mu_4}\,D^{\mu_1\,\mu_2}_{a_1\,a_2}) \\ & - \sum_{i,k\in S} I_4^{n+2}(i,k;S\setminus\{j\}) \\ & \times (\Delta^{\mu_1}_{a_1\,i}\,\Delta^{\mu_2}_{a_2\,k}\,D^{\mu_3\,\mu_4}_{a_3\,a_4} + \Delta^{\mu_1}_{a_1\,i}\,\Delta^{\mu_3}_{a_3\,k}\,D^{\mu_2\,\mu_4}_{a_2\,a_4} + \Delta^{\mu_1}_{a_1\,i}\,\Delta^{\mu_4}_{a_4\,k}\,D^{\mu_2\,\mu_3}_{a_2\,a_3} \\ & + \Delta^{\mu_2}_{a_2\,i}\,\Delta^{\mu_3}_{a_3\,k}\,D^{\mu_1\,\mu_4}_{a_1\,a_4} + \Delta^{\mu_2}_{a_2\,i}\,\Delta^{\mu_4}_{a_4\,k}\,D^{\mu_1\,\mu_3}_{a_1\,a_3} + \Delta^{\mu_3}_{a_3\,i}\,\Delta^{\mu_4}_{a_4\,k}\,D^{\mu_1\,\mu_2}_{a_1\,a_2})\Bigg] \end{aligned} \tag{C.106}$$

where

$$D^{\mu_1\,\mu_2}_{a_1\,a_2} = (b_j\,\mathcal{T}^{\mu_1\,\mu_2}_{a_1\,a_2} - \mathcal{C}^{\mu_2}_{j\,a_2}\,\mathcal{V}^{\mu_1}_{a_1} - \mathcal{C}^{\mu_1}_{j\,a_1}\,\mathcal{V}^{\mu_2}_{a_2}) \tag{C.107}$$

Similarly, eq. (C.95) leads to

$$Q_{2j}^{\mu_1} = (2-n)\sum_{i\in S}\Delta^{\mu_1}_{a_1\,i}\, I_4^{n+2}(i;S\setminus\{j\}) \tag{C.108}$$

$$\begin{aligned} Q_{2j}^{\mu_1\,\mu_2} = \; & (1-n)\sum_{j\in S}\Bigg[-\frac{1}{2}\, g^{\mu_1\,\mu_2}\, I_4^{n+4}(S\setminus\{j\}) \\ & + \sum_{i,k\in S}\Delta^{\mu_1}_{a_1\,i}\,\Delta^{\mu_2}_{a_2\,k}\, I_4^{n+2}(i,k;S\setminus\{j\})\Bigg] \end{aligned} \tag{C.109}$$

$$\begin{aligned} Q_{2j}^{\mu_1\,\mu_2\,\mu_3} = \; & n\Bigg[\frac{1}{2}\sum_{i\in S} I_4^{n+4}(i;S\setminus\{j\})\,(g^{\mu_1\,\mu_2}\,\Delta^{\mu_3}_{a_3\,i} + g^{\mu_1\,\mu_3}\,\Delta^{\mu_2}_{a_2\,i} + g^{\mu_2\,\mu_3}\,\Delta^{\mu_1}_{a_1\,i}) \\ & - \sum_{i,k,l\in S}\Delta^{\mu_1}_{a_1\,i}\,\Delta^{\mu_2}_{a_2\,k}\,\Delta^{\mu_3}_{a_3\,l}\, I_4^{n+2}(i,k,l;S\setminus\{j\})\Bigg] \end{aligned} \tag{C.110}$$

If we insert now eqs. (C.108) to (C.110) for Q_{2j} into the right-hand side of eq. (C.99) and use eq. (C.97), we obtain the expressions (C.104) to (C.106) for Q_1.

D Useful relations

In this appendix we give a collection of formulae which are useful if one wants to perform algebraic simplifications of loop amplitudes. The relations in subsection D.1 can also be very useful to perform checks on the implementation of the form factors in a computer program. Some of the relations are already given in the main text, but the purpose here is to list them for quick reference.

D.1 Relations between the form factors

The following identities have been used extensively to obtain the relations given below:

$$q_a \cdot \Delta_{b\,c} = \frac{1}{2}\left(q_b^2 - m_b^2 - [\, q_c^2 - m_c^2\,] - \mathcal{S}_{a\,b} + \mathcal{S}_{a\,c}\right) \tag{D.111}$$

$$\Delta_{l\,a} \cdot \Delta_{b\,c} = \frac{1}{2}\left(\mathcal{S}_{l\,c} - \mathcal{S}_{l\,b} + \mathcal{S}_{a\,b} - \mathcal{S}_{a\,c}\right) \tag{D.112}$$

Using relations (D.111) and (D.112) and multiplying with a vector Δ both the definition and the expression in terms of form factors of an integral, one finds relations between the form factors.

D.1.1 Four-point functions

For example, we have

$$\begin{aligned}
I_4^{n,\mu_1}(a;S)\,\Delta_{b\,c}^{\mu_1} &= \int d\bar{k}\,\frac{q_a\cdot\Delta_{b\,c}}{\prod_{i\in S}(q_i^2-m_i^2+i\delta)} \\
&= \frac{1}{2}\Bigg[\int d\bar{k}\,\frac{1}{\prod_{i\in S\setminus\{b\}}(q_i^2-m_i^2+i\delta)} - \int d\bar{k}\,\frac{1}{\prod_{i\in S\setminus\{c\}}(q_i^2-m_i^2+i\delta)} \\
&\quad -(\mathcal{S}_{a\,b}-\mathcal{S}_{a\,c})\int d\bar{k}\,\frac{1}{\prod_{i\in S}(q_i^2-m_i^2+i\delta)}\Bigg] \\
&= \frac{1}{2}\sum_{l\in S} A_l^{4,1}(S)\;(\,\mathcal{S}_{l\,c}-\mathcal{S}_{l\,b}+\mathcal{S}_{a\,b}-\mathcal{S}_{a\,c})
\end{aligned} \tag{D.113}$$

Eq. (D.113) implies

$$\sum_{l\in S} A_l^{4,1}(S) = -A^{4,0}(S) \tag{D.114}$$

$$\sum_{l\in S} A_l^{4,1}(S)\;(\,\mathcal{S}_{l\,c}-\mathcal{S}_{l\,b}) = A^{3,0}(S\setminus\{b\}) - A^{3,0}(S\setminus\{c\})\,. \tag{D.115}$$

One can proceed in the same way for all the four-point functions and finds for rank 2:

$$\sum_{l_2\in S} A_{l_1\,l_2}^{4,2}(S) = -A_{l_1}^{4,1}(S) \tag{D.116}$$

$$\begin{aligned}
\sum_{l_2\in S} A_{l_1\,l_2}^{4,2}(S)\;(\,\mathcal{S}_{l_2\,c}-\mathcal{S}_{l_2\,b}) &= 2\,(\delta_{l_1\,c}-\delta_{l_1\,b})\,B^{4,2}(S) \\
&\quad + \bar{\delta}_{l_1\,b}\,A_{l_1}^{3,1}(S\setminus\{b\}) - \bar{\delta}_{l_1\,c}\,A_{l_1}^{3,1}(S\setminus\{c\})\,,
\end{aligned} \tag{D.117}$$

where the definition

$$\bar{\delta}_{jl} = 1-\delta_{jl} = \begin{cases} 1 \text{ if } j\neq l \\ 0 \text{ if } j = l \end{cases} \tag{D.118}$$

has been used. For rank 3, one obtains:

$$\sum_{l\in S} B_l^{4,3}(S) = -B^{4,2}(S) \tag{D.119}$$

$$\sum_{l\in S} B_l^{4,3}(S)\,(\mathcal{S}_{l\,c}-\mathcal{S}_{l\,b}) = B^{3,2}(S\setminus\{b\}) - B^{3,2}(S\setminus\{c\}) \tag{D.120}$$

$$\sum_{l_3\in S} A_{l_1\,l_2\,l_3}^{4,3}(S) = -A_{l_1\,l_2}^{4,2}(S) \tag{D.121}$$

$$\begin{aligned}\sum_{l_3\in S} A_{l_1\,l_2\,l_3}^{4,3}(S)\,(\mathcal{S}_{l_3\,c}-\mathcal{S}_{l_3\,b}) &= 2\,(\delta_{l_2\,c}-\delta_{l_2\,b})\,B_{l_1}^{4,3}(S) + 2\,(\delta_{l_1\,c}-\delta_{l_1\,b})\,B_{l_2}^{4,3}(S) \\ &\quad + \bar{\delta}_{l_1\,b}\,\bar{\delta}_{l_2\,b}\,A_{l_1\,l_2}^{3,2}(S\setminus\{b\}) - \bar{\delta}_{l_1\,c}\,\bar{\delta}_{l_2\,c}\,A_{l_1\,l_2}^{3,2}(S\setminus\{c\})\end{aligned} \tag{D.122}$$

For rank 4:

$$\sum_{l_2\in S} B_{l_1\,l_2}^{4,4}(S) = -B_{l_1}^{4,3}(S) \tag{D.123}$$

$$\begin{aligned}\sum_{l_2\in S} B_{l_1\,l_2}^{4,4}(S)\,(\mathcal{S}_{l_2\,c}-\mathcal{S}_{l_2\,b}) &= 2\,(\delta_{l_1\,c}-\delta_{l_1\,b})\,C^{4,4}(S) \\ &\quad + \bar{\delta}_{l_1\,b}\,B_{l_1}^{3,3}(S\setminus\{b\}) - \bar{\delta}_{l_1\,c}\,B_{l_1}^{3,3}(S\setminus\{c\})\end{aligned} \tag{D.124}$$

$$\sum_{l_4\in S} A_{l_1\,l_2\,l_3\,l_4}^{4,4}(S) = -A_{l_1\,l_2\,l_3}^{4,3}(S) \tag{D.125}$$

$$\begin{aligned}\sum_{l_4\in S} A_{l_1\,l_2\,l_3\,l_4}^{4,4}(S)\,(\mathcal{S}_{l_4\,c}-\mathcal{S}_{l_4\,b}) &= 2\sum_{i=l_1,l_2,l_3}(\delta_{i\,c}-\delta_{i\,b})\,B_{\{l_1,l_2,l_3\}-\{i\}}^{4,4}(S) \\ &\quad + \bar{\delta}_{l_1\,b}\,\bar{\delta}_{l_2\,b}\,\bar{\delta}_{l_3\,b}\,A_{l_1\,l_2\,l_3}^{3,3}(S\setminus\{b\}) \\ &\quad - \bar{\delta}_{l_1\,c}\,\bar{\delta}_{l_2\,c}\,\bar{\delta}_{l_3\,c}\,A_{l_1\,l_2\,l_3}^{3,3}(S\setminus\{c\})\end{aligned} \tag{D.126}$$

D.1.2 Five-point functions

Below we will use the definition

$$H_{ij} = 2\left(\frac{b_i b_j}{B} - S_{ij}^{-1}\right) \tag{D.127}$$

Rank 1:

$$\sum_{l\in S} A_l^{5,1}(S) = -A^{5,0}(S) \tag{D.128}$$

$$\sum_{l\in S} A_l^{5,1}(S)\,(\mathcal{S}_{l\,c}-\mathcal{S}_{l\,b}) = A^{4,0}(S\setminus\{b\}) - A^{4,0}(S\setminus\{c\}) \tag{D.129}$$

Rank 2:

$$\sum_{l_2\in S} A_{l_1\,l_2}^{5,2}(S) = -A_{l_1}^{5,1}(S) \tag{D.130}$$

$$\begin{aligned}\sum_{l_2\in S} A_{l_1\,l_2}^{5,2}(S)\,(\mathcal{S}_{l_2\,c}-\mathcal{S}_{l_2\,b}) &= 2\,(\delta_{l_1\,c}-\delta_{l_1\,b})\,B^{5,2}(S) \\ &\quad + \bar{\delta}_{l_1\,b}\,A_{l_1}^{4,1}(S\setminus\{b\}) - \bar{\delta}_{l_1\,c}\,A_{l_1}^{4,1}(S\setminus\{c\})\end{aligned} \tag{D.131}$$

Rank 3:

$$\sum_{l\in S} B^{5,3}_{l}(S) = -B^{5,2}(S) \tag{D.132}$$

$$\sum_{l_3\in S} A^{5,3}_{l_1\,l_2\,l_3}(S) = -A^{5,2}_{l_1\,l_2}(S) \tag{D.133}$$

$$\begin{aligned}
&\sum_{l_3\in S} \left[A^{5,3}_{l_1\,l_2\,l_3}(S) + H_{l_1 l_2}\, B^{5,3}_{l_3}(S)\right]\, (\mathcal{S}_{l_3\,c} - \mathcal{S}_{l_3\,b}) = \\
&\quad H_{l_1 l_2}\, \left[B^{4,2}(S\setminus\{b\}) - B^{4,2}(S\setminus\{c\})\right] + 2\sum_{i=l_1,l_2} (\delta_{i\,c} - \delta_{i\,b})\, B^{5,3}_{\{l_1,l_2\}\setminus\{i\}}(S) \\
&\qquad + \bar{\delta}_{l_1\,b}\,\bar{\delta}_{l_2\,b}\, A^{4,2}_{l_1\,l_2}(S\setminus\{b\}) - \bar{\delta}_{l_1\,c}\,\bar{\delta}_{l_2\,c}\, A^{4,2}_{l_1\,l_2}(S\setminus\{c\})
\end{aligned} \tag{D.134}$$

Rank 4:

$$\sum_{l_2\in S} B^{5,4}_{l_1\,l_2}(S) = -B^{5,3}_{l_1}(S) \tag{D.135}$$

$$\sum_{l_4\in S} A^{5,4}_{l_1\,l_2\,l_3\,l_4}(S) = -A^{5,3}_{l_1\,l_2\,l_3}(S) \tag{D.136}$$

$$\begin{aligned}
&\sum_{l_4\in S} \left[A^{5,4}_{l_1\,l_2\,l_3\,l_4}(S) + H_{l_1 l_2}\, B^{5,4}_{l_3\,l_4}(S) + H_{l_1 l_3}\, B^{5,4}_{l_2\,l_4}(S) + H_{l_2 l_3}\, B^{5,4}_{l_1\,l_4}(S)\right]\, (\mathcal{S}_{l_4\,c} - \mathcal{S}_{l_4\,b}) = \\
&\quad H_{l_1 l_2}\, \left[2\,(\delta_{l_3\,c} - \delta_{l_3\,b})\, C^{5,4}(S) + \bar{\delta}_{l_3\,b}\, B^{4,3}_{l_3}(S\setminus\{b\}) - \bar{\delta}_{l_3\,c}\, B^{4,3}_{l_3}(S\setminus\{c\})\right] \\
&\quad + H_{l_1 l_3}\, \left[2\,(\delta_{l_2\,c} - \delta_{l_2\,b})\, C^{5,4}(S) + \bar{\delta}_{l_2\,b}\, B^{4,3}_{l_2}(S\setminus\{b\}) - \bar{\delta}_{l_2\,c}\, B^{4,3}_{l_2}(S\setminus\{c\})\right] \\
&\quad + H_{l_2 l_3}\, \left[2\,(\delta_{l_1\,c} - \delta_{l_1\,b})\, C^{5,4}(S) + \bar{\delta}_{l_1\,b}\, B^{4,3}_{l_1}(S\setminus\{b\}) - \bar{\delta}_{l_1\,c}\, B^{4,3}_{l_1}(S\setminus\{c\})\right] \\
&\quad + 2\sum_{i=l_1,l_2,l_3} (\delta_{i\,c} - \delta_{i\,b})\, B^{5,4}_{\{l_1,l_2,l_3\}\setminus\{i\}}(S) \\
&\quad + \bar{\delta}_{l_1\,b}\,\bar{\delta}_{l_2\,b}\,\bar{\delta}_{l_3\,b}\, A^{4,3}_{l_1\,l_2\,l_3}(S\setminus\{b\}) - \bar{\delta}_{l_1\,c}\,\bar{\delta}_{l_2\,c}\,\bar{\delta}_{l_3\,c}\, A^{4,3}_{l_1\,l_2\,l_3}(S\setminus\{c\})
\end{aligned} \tag{D.137}$$

Rank 5:

$$\sum_{l_5\in S} A^{5,5}_{l_1\,l_2\,l_3\,l_4\,l_5}(S) = -A^{5,4}_{l_1\,l_2\,l_3\,l_4}(S) \tag{D.138}$$

$$\begin{aligned}
& (H_{l_1\,l_2}\, H_{l_3\,l_4} + H_{l_1\,l_3}\, H_{l_2\,l_4} + H_{l_1\,l_4}\, H_{l_2\,l_3}) \sum_{l_5\in S} C^{5,5}_{l_5}(S) \\
& + H_{l_1\,l_2} \sum_{l_5\in S} B^{5,5}_{l_3\,l_4\,l_5}(S) + H_{l_1\,l_3} \sum_{l_5\in S} B^{5,5}_{l_2\,l_4\,l_5}(S) + H_{l_1\,l_4} \sum_{l_5\in S} B^{5,5}_{l_2\,l_3\,l_5}(S) \\
& + H_{l_2\,l_3} \sum_{l_5\in S} B^{5,5}_{l_1\,l_4\,l_5}(S) + H_{l_2\,l_4} \sum_{l_5\in S} B^{5,5}_{l_1\,l_3\,l_5}(S) + H_{l_3\,l_4} \sum_{l_5\in S} B^{5,5}_{l_1\,l_2\,l_5}(S) \\
= \ & -(H_{l_1\,l_2}\, H_{l_3\,l_4} + H_{l_1\,l_3}\, H_{l_2\,l_4} + H_{l_1\,l_4}\, H_{l_2\,l_3})\, C^{5,4}(S) \\
& - H_{l_1\,l_2}\, B^{5,4}_{l_3\,l_4}(S) - H_{l_1\,l_3}\, B^{5,4}_{l_2\,l_4}(S) - H_{l_1\,l_4}\, B^{5,4}_{l_2\,l_3}(S)
\end{aligned}$$

$$-H_{l_2\,l_3}\,B^{5,4}_{l_1\,l_4}(S)-H_{l_2\,l_4}\,B^{5,4}_{l_1\,l_3}(S)-H_{l_3\,l_4}\,B^{5,4}_{l_1\,l_2}(S) \tag{D.139}$$

$$\begin{aligned}
&(H_{l_1\,l_2}\,H_{l_3\,l_4}+H_{l_1\,l_3}\,H_{l_2\,l_4}+H_{l_1\,l_4}\,H_{l_2\,l_3})\sum_{l_5\in S}(\mathcal{S}_{l_5\,c}-\mathcal{S}_{l_5\,b})\,C^{5,5}_{l_5}(S)\\
&+H_{l_1\,l_2}\left[2\,(\delta_{l_3\,b}-\delta_{l_3\,c})\,C^{5,5}_{l_4}(S)+2\,(\delta_{l_4\,b}-\delta_{l_4\,c})\,C^{5,5}_{l_3}(S)+\sum_{l_5\in S}(\mathcal{S}_{l_5\,c}-\mathcal{S}_{l_5\,b})\,B^{5,5}_{l_3\,l_4\,l_5}(S)\right]\\
&+H_{l_1\,l_3}\left[2\,(\delta_{l_2\,b}-\delta_{l_2\,c})\,C^{5,5}_{l_4}(S)+2\,(\delta_{l_4\,b}-\delta_{l_4\,c})\,C^{5,5}_{l_2}(S)+\sum_{l_5\in S}(\mathcal{S}_{l_5\,c}-\mathcal{S}_{l_5\,b})\,B^{5,5}_{l_2\,l_4\,l_5}(S)\right]\\
&+H_{l_1\,l_4}\left[2\,(\delta_{l_2\,b}-\delta_{l_2\,c})\,C^{5,5}_{l_3}(S)+2\,(\delta_{l_3\,b}-\delta_{l_3\,c})\,C^{5,5}_{l_2}(S)+\sum_{l_5\in S}(\mathcal{S}_{l_5\,c}-\mathcal{S}_{l_5\,b})\,B^{5,5}_{l_2\,l_3\,l_5}(S)\right]\\
&+H_{l_2\,l_3}\left[2\,(\delta_{l_1\,b}-\delta_{l_1\,c})\,C^{5,5}_{l_4}(S)+2\,(\delta_{l_4\,b}-\delta_{l_4\,c})\,C^{5,5}_{l_1}(S)+\sum_{l_5\in S}(\mathcal{S}_{l_5\,c}-\mathcal{S}_{l_5\,b})\,B^{5,5}_{l_1\,l_4\,l_5}(S)\right]\\
&+H_{l_2\,l_4}\left[2\,(\delta_{l_1\,b}-\delta_{l_1\,c})\,C^{5,5}_{l_3}(S)+2\,(\delta_{l_3\,b}-\delta_{l_3\,c})\,C^{5,5}_{l_1}(S)+\sum_{l_5\in S}(\mathcal{S}_{l_5\,c}-\mathcal{S}_{l_5\,b})\,B^{5,5}_{l_1\,l_3\,l_5}(S)\right]\\
&+H_{l_3\,l_4}\left[2\,(\delta_{l_1\,b}-\delta_{l_1\,c})\,C^{5,5}_{l_2}(S)+2\,(\delta_{l_2\,b}-\delta_{l_2\,c})\,C^{5,5}_{l_1}(S)+\sum_{l_5\in S}(\mathcal{S}_{l_5\,c}-\mathcal{S}_{l_5\,b})\,B^{5,5}_{l_1\,l_2\,l_5}(S)\right]\\
&+\sum_{l_5\in S}(\mathcal{S}_{l_5\,c}-\mathcal{S}_{l_5\,b})\,A^{5,5}_{l_1\,l_2\,l_3\,l_4\,l_5}(S)\\
&+2\,(\delta_{l_1\,b}-\delta_{l_1\,c})\,B^{5,5}_{l_2\,l_3\,l_4}(S)+2\,(\delta_{l_2\,b}-\delta_{l_2\,c})\,B^{5,5}_{l_1\,l_3\,l_4}(S)\\
&+2\,(\delta_{l_3\,b}-\delta_{l_3\,c})\,B^{5,5}_{l_1\,l_2\,l_4}(S)+2\,(\delta_{l_4\,b}-\delta_{l_4\,c})\,B^{5,5}_{l_1\,l_2\,l_3}(S)\\
=\;&(H_{l_1\,l_2}\,H_{l_3\,l_4}+H_{l_1\,l_3}\,H_{l_2\,l_4}+H_{l_1\,l_4}\,H_{l_2\,l_3})\,(C^{4,4}(S\setminus\{b\})-C^{4,4}(S\setminus\{c\}))\\
&+H_{l_1\,l_2}\,(\bar\delta_{l_3\,b}\,\bar\delta_{l_4\,b}\,B^{4,4}_{l_3\,l_4}(S\setminus\{b\})-\bar\delta_{l_3\,c}\,\bar\delta_{l_4\,c}\,B^{4,4}_{l_3\,l_4}(S\setminus\{c\}))\\
&+H_{l_1\,l_3}\,(\bar\delta_{l_2\,b}\,\bar\delta_{l_4\,b}\,B^{4,4}_{l_2\,l_4}(S\setminus\{b\})-\bar\delta_{l_2\,c}\,\bar\delta_{l_4\,c}\,B^{4,4}_{l_2\,l_4}(S\setminus\{c\}))\\
&+H_{l_1\,l_4}\,(\bar\delta_{l_2\,b}\,\bar\delta_{l_3\,b}\,B^{4,4}_{l_2\,l_3}(S\setminus\{b\})-\bar\delta_{l_2\,c}\,\bar\delta_{l_3\,c}\,B^{4,4}_{l_2\,l_3}(S\setminus\{c\}))\\
&+H_{l_2\,l_3}\,(\bar\delta_{l_1\,b}\,\bar\delta_{l_4\,b}\,B^{4,4}_{l_1\,l_4}(S\setminus\{b\})-\bar\delta_{l_1\,c}\,\bar\delta_{l_4\,c}\,B^{4,4}_{l_1\,l_4}(S\setminus\{c\}))\\
&+H_{l_2\,l_4}\,(\bar\delta_{l_1\,b}\,\bar\delta_{l_3\,b}\,B^{4,4}_{l_1\,l_3}(S\setminus\{b\})-\bar\delta_{l_1\,c}\,\bar\delta_{l_3\,c}\,B^{4,4}_{l_1\,l_3}(S\setminus\{c\}))\\
&+H_{l_3\,l_4}\,(\bar\delta_{l_1\,b}\,\bar\delta_{l_2\,b}\,B^{4,4}_{l_1\,l_2}(S\setminus\{b\})-\bar\delta_{l_1\,c}\,\bar\delta_{l_2\,c}\,B^{4,4}_{l_1\,l_2}(S\setminus\{c\}))\\
&+\bar\delta_{l_1\,b}\,\bar\delta_{l_2\,b}\,\bar\delta_{l_3\,b}\,\bar\delta_{l_4\,b}\,A^{4,4}_{l_1\,l_2\,l_3\,l_4}(S\setminus\{b\})-\bar\delta_{l_1\,c}\,\bar\delta_{l_2\,c}\,\bar\delta_{l_3\,c}\,\bar\delta_{l_4\,c}\,A^{4,4}_{l_1\,l_2\,l_3\,l_4}(S\setminus\{c\})
\end{aligned} \tag{D.140}$$

D.2 Relations between reduction coefficients

The following relation is useful to cancel $1/B$ terms in the form factors for 5-point integrals:

$$\mathcal{S}^{-1}_{c\,l}\,\left(b^{\{c\}}_j-b_j\right)-b_c\,\left(\mathcal{S}^{\{c\}-1}{}_{j\,l}-\mathcal{S}^{-1}_{j\,l}\right)\;=\;0 \tag{D.141}$$

To prove eq. (D.141), we introduce the auxiliary relation

$$-\,(\mathcal{S}^{\{c\}})^{-1}_{j\,l}+\mathcal{S}^{-1}_{j\,l}+\mathcal{S}^{-1}_{c\,l}\,\mathcal{Y}_{j\,c}=0\;, \tag{D.142}$$

$$\text{where}\quad \mathcal{Y}_{j\,c} = \sum_{l\in S\setminus\{c\}} (\mathcal{S}^{\{c\}})^{-1}_{j\,l}\,\mathcal{S}_{l\,c}$$

To show relation (D.142), let us assume the right-hand side is not zero, but some tensor α_{jlc}:

$$-\,\mathcal{S}^{\{c\}-1}{}_{j\,l} + \mathcal{S}^{-1}{}_{j\,l} + \mathcal{S}^{-1}{}_{c\,l}\,\mathcal{Y}_{j\,c} = \alpha_{jlc} \tag{D.143}$$

Now, we compute

$$\begin{aligned}\sum_{l\in S}\mathcal{S}_{k\,l}\,\alpha_{jlc} &= -\sum_{l\in S\setminus\{c\}}\mathcal{S}_{k\,l}\,\mathcal{S}^{\{c\}-1}{}_{j\,l} + \delta_{k\,j} + \delta_{c\,k}\,\mathcal{Y}_{j\,c}\\ &= \begin{cases} -\delta_{j\,k} + \delta_{j\,k} = 0 & \text{if } k\neq c\\ -\mathcal{Y}_{j\,c} + \mathcal{Y}_{j\,c} = 0 & \text{if } k = c\end{cases}\end{aligned}$$

Since $\mathcal{S}$ is invertible, we must have $\alpha_{jlc} = 0$. Summing eq. (D.142) over $l\in S$ yields

$$-\,b_j^{\{c\}} + b_j + b_c\,\mathcal{Y}_{j\,c} = 0 \tag{D.144}$$

Now we multiply eq. (D.142) with b_c and eq. (D.144) with $\mathcal{S}^{-1}_{cl}$ and take the difference of the two resulting equations to obtain eq. (D.141).
Multiplying eq. (D.141) by $\Delta^\mu_{l\,a}$ and summing over $l\in S$ leads to

$$\mathcal{C}^\mu_{c\,a}\,b_j^{\{c\}} - b_c\,\mathcal{C}^{\{c\}}{}^\mu_{j\,a} = \mathcal{C}^\mu_{c\,a}\,b_j - b_c\,\mathcal{C}^\mu_{j\,a} \tag{D.145}$$

Summing eq. (D.145) over j in $S\setminus\{c\}$ yields

$$\mathcal{C}^\mu_{c\,a}\,B^{\{c\}} - b_c\,\mathcal{V}^{\{c\}}{}^\mu_a = \mathcal{C}^\mu_{c\,a}\,B - b_c\,\mathcal{V}^\mu_a \tag{D.146}$$

and summing eq. (D.145), multiplied by Δ^ν_{jb}, over j in $S\setminus\{c\}$, we obtain

$$\mathcal{C}^\mu_{c\,a}\,\mathcal{V}^{\{c\}}{}^\nu_b - \frac{1}{2}\,b_c\,\mathcal{T}^{\{c\}}_{[4]}{}^{\mu\,\nu}_{a\,b} = \mathcal{C}^\mu_{c\,a}\,\mathcal{V}^\nu_b - \frac{1}{2}\,b_c\,\mathcal{T}_{[4]}{}^{\mu\,\nu}_{a\,b}\;. \tag{D.147}$$

Further, one can show by direct calculation:

$$\sum_{i\in S} b_i\,(\Delta^2_{i\,a} - m_i^2) \;=\; 1 + B\,m_a^2 \tag{D.148}$$

$$\sum_{i,j\in S}\Delta^\mu_{i\,a}\,\mathcal{S}^{-1}_{ij}\,\Delta^2_{j\,c} \;=\; \Delta^\mu_{c\,a} + m_c^2\,\mathcal{V}^\mu_a + \sum_{j\in S} m_j^2\,\mathcal{C}^\mu_{j\,a} \tag{D.149}$$

D.3 Special relations for $N = 5$

For $N = 5$, the external vectors form a basis of Minkowski space, such that the metric (in 4 dimensions) can be expressed by the tensor $\mathcal{H}^{\mu\nu}$, constructed from external vectors only:

$$\mathcal{H}^{\mu\nu}_{ab} \;=\; \sum_{i,j\in S}\Delta^\mu_{i\,a}\left(\frac{b_i b_j}{B} - \mathcal{S}^{-1}_{ij}\right)\Delta^\nu_{j\,b} = \frac{g^{\mu\nu}_{[4]}}{2}\;, \tag{D.150}$$

This fact implies the relation

$$\mathcal{T}_{[4]}{}^{\mu\,\nu}_{a\,b} = 2\,\frac{\mathcal{V}^\mu_a\,\mathcal{V}^\nu_b}{B} \quad \text{for } N=5\ \ . \tag{D.151}$$

The proof is straightforward:
Multiply the definition of $\mathcal{H}^{\mu\nu}_{ab}$ by $\Delta^\mu_{il}\,\Delta^\nu_{mn}$, we obtain

$$\mathcal{H}^{\mu\nu}_{ab}\,\Delta^\mu_{i\,l}\,\Delta^\nu_{m\,n} \;\; = \;\; \sum_{j,k\in S}\left[\frac{b_j\,b_k}{B} - \mathcal{S}^{-1}_{jk}\right]\,\Delta_{j\,a}\cdot\Delta_{i\,l}\,\Delta_{k\,b}\cdot\Delta_{m\,n} \tag{D.152}$$

Using now

$$\Delta_{j\,a}\cdot\Delta_{i\,c} = \frac{1}{2}\,(\,\mathcal{S}_{j\,c} - \mathcal{S}_{j\,i} + \mathcal{S}_{a\,i} - \mathcal{S}_{a\,c})$$

we get:

$$\mathcal{H}^{\mu\nu}_{ab}\,\Delta^\mu_{il}\,\Delta^\nu_{mn} = \frac{1}{2}\,\Delta_{il}\cdot\Delta_{mn} = \frac{1}{2}\,g^{\mu\nu}_{[4]}\,\Delta^\mu_{il}\,\Delta^\nu_{mn} \tag{D.153}$$

As the vectors Δ form a basis of Minkowski space for $N=5$, we conclude that

$$\mathcal{H}^{\mu\nu}_{ab} = \frac{1}{2}\,g^{\mu\nu}_{[4]}\ . \tag{D.154}$$

On the other hand, using the definitions (see eqs. (53) and (51))

$$\begin{aligned}
\mathcal{T}^{\mu\,\nu}_{a\,b} &= g^{\mu\,\nu} + 2\sum_{j,k\in S}\mathcal{S}^{-1}{}_{j\,k}\,\Delta^\mu_{k\,a}\,\Delta^\nu_{j\,b} \\
\mathcal{V}^\mu_a &= \sum_{k\in S} b_k\,\Delta^\mu_{k\,a}
\end{aligned}$$

we see that $\mathcal{H}^{\mu\nu}_{ab}$ is equal to

$$\mathcal{H}^{\mu\nu}_{ab} = \frac{\mathcal{V}^\mu_a\,\mathcal{V}^\nu_b}{B} - \frac{1}{2}\left(\mathcal{T}_{[4]}{}^{\mu\nu}_{a\,b} - g^{\mu\nu}_{[4]}\right) \tag{D.155}$$

such that, together with eq. (D.154), we must have

$$\mathcal{T}_{[4]}{}^{\mu\,\nu}_{a\,b} = 2\,\frac{\mathcal{V}^\mu_a\,\mathcal{V}^\nu_b}{B} \quad \text{for } N=5\ . \tag{D.156}$$

D.4 Special relations for $N = 6$

$$B = \sum_{i=1}^{6} b_i \;\; = \;\; 0 \tag{D.157}$$

$$\sum_{i=1}^{6} b_i\,\Delta^\mu_{i\,a} \;\; = \;\; \mathcal{V}^\mu_a = 0 \tag{D.158}$$

$$\sum_{i=1}^{6} b_i\,(q_i^2 - m_i^2) \;\; = \;\; 1 \tag{D.159}$$

$$\sum_{i,j=1}^{6}\,\Delta^\mu_{i\,a}\,\mathcal{S}^{-1}_{ij}\Delta^\nu_{j\,b} \;\; = \;\; \sum_{j=1}^{6}\,\mathcal{C}^\mu_{j\,a}\,\Delta^\nu_{j\,b} = -\frac{1}{2}\,g^{\mu\nu}_{[4]} \tag{D.160}$$

E Hexagon relations from helicity decomposition

In the present appendix we provide very compact expressions for the reduction coefficients b_i and $\mathcal{C}^{\mu}_{ia}$ in the specific six-point case where the particles in the loop as well as all legs are massless.

For massless gauge theory amplitudes usually the spinor helicity formalism [83, 84, 85, 86, 87] is used for the treatment of vector bosons as well as for massless fermions. In this formalism the spinor degrees of freedom are systematically projected on helicity eigenstates. This is often essential if one wishes to write down the final result for an amplitude in a compact way. Now, one of the technical points which had emerged in the massless six-point calculations of [71, 72] is that the reduction coefficients b_i can be written very compactly in terms of spinor traces, e.g.

$$b_1 = \frac{(123456)(3456) - 2s_{34}s_{45}s_{56}(6123)}{\det \mathcal{S}} \tag{E.161}$$

$$\det \mathcal{S} = 4s_{12}s_{23}s_{34}s_{45}s_{56}s_{61} - (123456)^2 \, . \tag{E.162}$$

Here and in the following $(12\cdots)$ is a shorthand notation for $\mathrm{tr}\,(\not p_1 \not p_2 \cdots)$. The corresponding formulae for $b_2, \ldots, b_6$ are obtained by cyclic permutation.

In the present appendix we first show that the helicity decomposition leads to further simplifications in eqs. (E.161), (E.162) and yields even more compact expressions. The two main identities responsible for the surprising simplifications in the Yukawa model calculation of [71] become quite transparent in this approach. More surprising is perhaps that, even for the plain six-point determinant $\det \mathcal{S}$, the introduction of chirality leads to a simple formula for its square root which cannot be written in terms of the Mandelstam variables alone. We present this further simplification of eqs. (E.161), (E.162) in eqs. (E.165), (E.168) below. The helicity representation also yields very compact expressions for the reduction coefficients $\mathcal{C}^{\mu}_{ia}$.

We use here the helicity representation only as a means to get more compact formulae than those provided by the general method described in the main text for the fully massless $N = 6$ case. Before we proceed let us however stress that the helicity representation naturally leads to an alternative algorithm for the reduction of fully massless six-point rank r integrals, still into a linear combination of five-point rank $r-1$ integrals, though decomposed on a different tensor basis. In this appendix we only sketch how this algorithm proceeds for the six point rank one tensor integrals. We then match the reduction thus obtained with the one derived in section 4.2, which provides us with the above mentioned compact expressions for the reduction coefficients $\mathcal{C}^{\mu}_{ia}$.

Let us introduce the notation[8]

$$\begin{aligned}
(+ijk\cdots s) &\equiv \mathrm{tr}\left(\frac{1}{2}(\mathbb{1}+\gamma_5)\not p_i \not p_j \not p_k \cdots \not p_s\right) = [ij]\langle jk\rangle[k\cdots s]\langle si\rangle \\
(-ijk\cdots s) &\equiv \mathrm{tr}\left(\frac{1}{2}(\mathbb{1}-\gamma_5)\not p_i \not p_j \not p_k \cdots \not p_s\right) = \langle ij\rangle[jk]\langle k\cdots s\rangle[si]
\end{aligned} \tag{E.163}$$

The product of two oppositely handed traces containing the same substring of some momenta $p_{i_1}\cdots p_{i_l}$ obeys the useful identity

$$(+i_1 i_2 \cdots i_l j_1 j_2 \cdots j_s)(-i_1 i_2 \cdots i_l k_1 k_2 \cdots k_t)$$

[8] Our spinor helicity conventions follow [88]; in particular $\mathrm{tr}\,(\gamma_5 ijkl) = 4i\varepsilon(ijkl)$.

$$= s_{i_1 i_2} s_{i_2 i_3} \cdots s_{i_{l-1} i_l} ((-)^l j_1 j_2 \cdots j_s i_1 k_t k_{t-1} \cdots k_1 i_l) \tag{E.164}$$

We can apply this to rewrite $\det \mathcal{S}$ as follows:

$$\begin{aligned} \det \mathcal{S} &= 4 s_{12} s_{23} s_{34} s_{45} s_{56} s_{61} - (123456)^2 \\ &= 4(+123456)(-123456) - \Big((+123456) + (-123456)\Big)^2 \\ &= -\Big((+123456) - (-123456)\Big)^2 = -(\gamma_5 123456)^2 \end{aligned} \tag{E.165}$$

Similarly, in eq. (E.161) the numerator of, say, b_5 can be written as

$$\begin{aligned} (123456)(1234) - 2 s_{12} s_{23} s_{34} (4561) &= \Big((+123456) + (-123456)\Big)\Big((+1234) + (-1234)\Big) \\ &\quad -2(+123456)(-1234) - 2(-123456)(+1234) \\ &= \Big((+123456) - (-123456)\Big)\Big((+1234) - (-1234)\Big) \end{aligned} \tag{E.166}$$

Combining this result with eq.(E.165) one factor of $(+123456) - (-123456)$ cancels between numerator and denominator, and b_5 becomes simply

$$b_5 = -\frac{(+1234) - (-1234)}{(+123456) - (-123456)} = -\frac{(\gamma_5 1234)}{(\gamma_5 123456)} = -4i \frac{\varepsilon(1234)}{(\gamma_5 123456)} \tag{E.167}$$

By cyclicity,

$$b_i = (-1)^i 4i \, \frac{\varepsilon((i+2)(i+3)(i+4)(i+5))}{(\gamma_5 123456)} \tag{E.168}$$

The redundancy relation $\sum_{i=1}^{6} b_i = 0$ (eq. (D.157)) is straightforwardly checked by the chiral representation (E.168) for the reduction coefficients b_i, as it now reads:

$$\varepsilon(1234) - \varepsilon(2345) + \varepsilon(3456) - \varepsilon(4561) + \varepsilon(5612) - \varepsilon(6123) = 0 \tag{E.169}$$

which is ensured trivially by momentum conservation[9].

Next, let us recall the two main identities used in [71] for simplifying the calculation of the six-scalar Yukawa model amplitude:

$$(3456) + (123456) b_1 = -2 s_{34} s_{45} s_{56} b_4 \tag{E.170}$$

$$(14) + b_2(1234) + b_5(4561) = 0 \tag{E.171}$$

Eq. (E.170) is proved as follows:

$$(3456) + (123456)\, b_1 = \frac{[(+3456) + (-3456)][(+123456) - (-123456)]}{(+123456) - (-123456)}$$

[9]It is *not* meant here that eq. (D.157) is the consequence of momentum conservation. On the contrary, it expresses the extra constraint that the Gram determinant vanishes for $N = 6$ [46, 50] i.e. that any five of the six external momenta p_i, $i = 1, \cdots, 6$ are linearly dependent in a four-dimensional space-time.

$$
\begin{aligned}
& -\frac{[(+123456)+(-123456)][(+3456)-(-3456)]}{(+123456)-(-123456)} \\
= \;& 2\frac{(+345612)(-3456)-(-345612)(+3456)}{(+123456)-(-123456)} \\
= \;& 2s_{34}s_{45}s_{56}\frac{(+1236)-(-1236)}{(+123456)-(-123456)} \\
= \;& -2s_{34}s_{45}s_{56}\, b_4
\end{aligned}
\tag{E.172}
$$

Here (E.164) was used in the third step. The proof of (E.171) is similar.

The representation (E.168) for b_i further suggests that, for the reduction of six-point tensor integrals, it might be natural to expand one of the numerator momenta in terms of vectors dual to the external momenta along the lines of [40]. Let us define a dual basis $\{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, v_4^{(i)}\}$, $i = 1, \ldots, 6$, for each of the six choices of four consecutive momenta[10]:

$$
\begin{aligned}
v_{1\mu}^{(i)} &= (-1)^i \varepsilon(\mu, i+3, i+4, i+5), \\
v_{2\mu}^{(i)} &= (-1)^i \varepsilon(i+2, \mu, i+4, i+5), \\
v_{3\mu}^{(i)} &= (-1)^i \varepsilon(i+2, i+3, \mu, i+5), \\
v_{4\mu}^{(i)} &= (-1)^i \varepsilon(i+2, i+3, i+4, \mu)
\end{aligned}
$$

For generic external momenta, each of these six bases can be used to expand the loop momentum k. Namely, further defining

$$
\varepsilon^{(i)} \equiv (-1)^i \varepsilon(i+2, i+3, i+4, i+5) \tag{E.173}
$$

one has

$$
v_a^{(i)} \cdot p_k = \varepsilon^{(i)} \delta_{k,i+a+1} \qquad (k = i+2, \ldots, i+5) \tag{E.174}
$$

Therefore,

$$
q_{a_m}^\mu = \frac{1}{\varepsilon^{(i)}} \sum_{a=1}^{4} v_a^{(i)\mu}\, p_{i+a+1} \cdot q_{a_m} \qquad (i = 1, \ldots, 6) \tag{E.175}
$$

The only caveat here is that we have used a projection of the momentum $q_{a_m}^\mu$ in the loop on four-dimensional space, while the loop integration requires dimensional continuation. However, for rank one integrals this is safe, since at one loop the index μ will be contracted with some external momentum or polarisation only. We will comment briefly on the case of higher ranks below.

Using this equation together with eq. (D.159) and rewriting

$$
p_{i+a+1} \cdot q_{a_m} = \frac{1}{2}\Big[q_{i+a+1}^2 - q_{i+a}^2 - \mathcal{S}_{i+a+1,a_m} + \mathcal{S}_{i+a,a_m}\Big] \tag{E.176}
$$

[10] Note that contrary to [40] we use the p_i rather than the r_i in building the dual basis vectors.

we obtain

$$
\begin{aligned}
q^{\mu}_{a_m} &= \sum_{i=1}^{6} b_i q_i^2 \frac{1}{\varepsilon^{(i)}} \sum_{a=1}^{4} v_a^{(i)\mu}\, p_{i+a+1}\cdot q_{a_m} \\
&= \frac{2i}{(\gamma_5 123456)} \sum_{i=1}^{6} q_i^2 \sum_{a=1}^{4} \Big[q^2_{i+a+1} - q^2_{i+a} - \mathcal{S}_{i+a+1,a_m} + \mathcal{S}_{i+a,a_m} \Big]\, v_a^{(i)\mu} \qquad \text{(E.177)}
\end{aligned}
$$

Note that the $\varepsilon^{(i)}$ from the numerator of b_i in eq. (E.168) has cancelled against the same $\varepsilon^{(i)}$ from the denominator of eq.(E.175); this is the main point of the algorithm.

The $v_a^{(i)}$ are not independent since

$$
v_1^{(i)} = v_4^{(i+1)} \qquad \text{(E.178)}
$$

and, by momentum conservation, also

$$
\begin{aligned}
v_1^{(i)} - v_2^{(i)} &= -(v_3^{(i+2)} - v_4^{(i+2)}) \qquad \text{(E.179)} \\
v_2^{(i)} - v_3^{(i)} &= -(v_2^{(i+3)} - v_3^{(i+3)}) \qquad \text{(E.180)}
\end{aligned}
$$

Remarkably, the three identities (E.178),(E.179),(E.180) together just imply that the 48 products of q_i^2's in (E.177) cancel out in pairs. This leaves us with

$$
q^{\mu}_{a_m} = \frac{2i}{(\gamma_5 123456)} \sum_{i=1}^{6} q_i^2 \sum_{a=1}^{4} \Big(\mathcal{S}_{i+a,a_m} - \mathcal{S}_{i+a+1,a_m} \Big)\, v_a^{(i)\mu} \qquad \text{(E.181)}
$$

Using the decomposition (E.181) in the rank one massless six-point integral $I_6^{n,\mu}$ we obtain the following reduction to five-point rank zero integrals:

$$
I_6^{n,\mu}(a_m; S) = \frac{2i}{(\gamma_5 123456)} \sum_{i=1}^{6} \sum_{a=1}^{4} \Big(\mathcal{S}_{i+a,a_m} - \mathcal{S}_{i+a+1,a_m} \Big)\, v_a^{(i)\mu}\, I_5^n(S \setminus \{i\}) \qquad \text{(E.182)}
$$

We can write this result more compactly by introducing the convention that $v_5^{(i)\mu} = v_6^{(i)\mu} = 0$, and defining

$$
d_a^{(i)\mu} \equiv v_{a-i-1}^{(i)\mu} - v_{a-i}^{(i)\mu} \qquad (i, a = 1, \ldots, 6) \qquad \text{(E.183)}
$$

In terms of the vectors $d_a^{(i)\mu}$, eqs. (E.178),(E.179),(E.180) can be neatly combined to

$$
d_j^{(i)} = -\, d_i^{(j)} \qquad (i, j = 1, \ldots, 6) \qquad \text{(E.184)}
$$

We also note that

$$
\begin{aligned}
\sum_{a=1}^{6} d_a^{(i)\mu} &= 0 \qquad (i = 1, \ldots, 6) \qquad \text{(E.185)} \\
\sum_{j=1}^{6} r_j^{\mu} d_j^{(i)\nu} &= \varepsilon^{(i)}\, g^{\mu\nu}_{[4]} \qquad (i = 1, \ldots, 6) \qquad \text{(E.186)}
\end{aligned}
$$

Now, further defining

$$\mathcal{D}^{\mu}_{ia_m} \equiv \sum_{a=1}^{6} \mathcal{S}_{a\,a_m}\, d_a^{(i)\mu} \qquad (i, a_m = 1, \ldots, 6) \tag{E.187}$$

one has

$$I_6^{n,\mu}(a_m; S) = -\frac{2i}{(\gamma_5 123456)} \sum_{i=1}^{6} \mathcal{D}^{\mu}_{ia_m}\, I_5^n(S \setminus \{i\}) \tag{E.188}$$

Comparing eq. (E.188) with eq. (46) of section 4.2 for rank one – in which the first term on the r.h.s. vanishes as there is no higher dimensional integral for $N = 6$ – we identify

$$\mathcal{C}^{\mu}_{ia} = \frac{2i}{(\gamma_5 123456)}\, \mathcal{D}^{\mu}_{ia} \tag{E.189}$$

Let us add one comment on the application of this algorithm in the case of arbitrary rank. Applying the decomposition (E.181) to the six-point integral of arbitrary rank we obtain

$$I_6^{n,\mu_1,\ldots,\mu_r}(a_1, \ldots, a_r; S) = -\frac{2i}{(\gamma_5 123456)} \sum_{i=1}^{6} \mathcal{D}^{\mu_1}_{ia_1}\, I_5^{n,\mu_2,\cdots,\mu_r}(a_2, \ldots, a_r; S \setminus \{i\}) \tag{E.190}$$

In the case of arbitrary rank the implied projection of the loop momentum $q^{\mu}_{a_m}$ on four-dimensional space requires a more careful consideration. This trick could fail if a contraction of two indices with a metric tensor $g_{\mu\nu}$ occurs, as can be the case at ranks $r \geq 2$. However, if $g_{\mu\nu}$ is the full n-dimensional metric, then the integral breaks down to a lower rank integral anyway, while if $g_{\mu\nu}$ is the $(n-4)$-dimensional metric then $g_{[n-4]} = g_{[n]} - g_{[4]}$ can be used before applying (E.190).

In any case the index "μ" gets absorbed into the vectors $d_a^{(i)\mu}$, and ultimately will be contracted either (i) with an external momentum $p_{l\mu}$, (ii) with an ε - tensor, or (iii) with a polarisation vector $\varepsilon^{\pm}_{l\mu}(p_k, q)$. Let us consider these cases in turn.
(i) Contraction with external momenta produces ε-tensors. These can, using momentum conservation, be reduced to the six $\varepsilon^{(i)}$'s.
(ii) Here we encounter the contraction of two ε-tensors involving arbitrary momenta,

$$\begin{aligned}
\varepsilon(\mu, a, b, c)\, \varepsilon(\mu, i, j, k) &= \det \begin{pmatrix} p_a \cdot p_i & p_a \cdot p_j & p_a \cdot p_k \\ p_b \cdot p_i & p_b \cdot p_j & p_b \cdot p_k \\ p_c \cdot p_i & p_c \cdot p_j & p_c \cdot p_k \end{pmatrix} \\
&= \frac{1}{8}\Big[(abckji) - (abcijk)\Big]
\end{aligned} \tag{E.191}$$

(iii) As usual we take the reference momentum for ε_l to be some other external momentum p_s, so that

$$\varepsilon^{\pm}_{l\mu}(p_s) = \pm \frac{\langle s^{\mp}|\gamma_\mu|l^{\mp}\rangle}{\sqrt{2}\langle s^{\mp}|l^{\pm}\rangle} \tag{E.192}$$

Then it is easily shown by using the Fierz identity that

$$\begin{aligned}
\varepsilon^+_{l\mu}(p_s)\,\varepsilon(\mu,i,j,k) &= \frac{i}{2\sqrt{2}\langle ls\rangle}\Big[\frac{\langle ks\rangle}{\langle kl\rangle}(+lijk) - \frac{\langle is\rangle}{\langle il\rangle}(-lijk)\Big] \\
\varepsilon^-_{l\mu}(p_s)\,\varepsilon(\mu,i,j,k) &= \frac{i}{2\sqrt{2}[ls]}\Big[\frac{[ks]}{[kl]}(-lijk) - \frac{[is]}{[il]}(+lijk)\Big]
\end{aligned} \tag{E.193}$$

In writing eqs. (E.193) we have assumed $i, k \neq l$, which is not a restriction.

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